Regularity properties of solutions to systems describing non-Newtonian fluids

(Habilitation Thesis)

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Regularita řešení systémů popisujících nenewtonovské proudění

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I would like to express my gratitude to everyone who has supported me on my journey to where I am now. In addition to my family, I am also deeply grateful to my coworkers, whose patience and tolerance were immeasurable.

This habilitation thesis is based on articles [1], [2], [3], [4], [5] and [6] where the analysis of the flow of non-Newtonian fluid is provided. In particular, the flow of a compressible (barotropic) fluid is described by the system

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) &= 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) &= \operatorname{div} \mathbb{S} \end{aligned} \tag{1}$$

which is considered on a time-space domain $(0, T) \times \Omega$, $T > 0$, $\Omega \in \mathbb{R}^n$. It is assumed that the flow is entirely described by two unknown variables ϱ, which stands for the density of the fluid, and u, which denotes fluid's velocity. The pressure p is usually assumed to be a sufficiently smooth, increasing function, and the stress tensor $\mathbb{S} = \mathbb{S}(\varrho, \nabla u)$ is a given function taking values in the space of $n \times n$ real matrices.

In the case where $\mathbb{S}$ takes the form $\mathbb{S} = \mu Du$, $\mu > 0$, $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$, the fluid behaves in a Newtonian manner. However, many fluids exhibit non-Newtonian behavior, where the stress tensor does not provide a linear response to the velocity gradient. See for instance [7] or [8] and the references therein.

For incompressible fluid, where $\operatorname{div} u = 0$, the flow is described by

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla \pi &= \operatorname{div} \mathbb{S} \\ \operatorname{div} u &= 0 \end{aligned} \tag{2}$$

with unknowns u (velocity) and π (pressure). This system can be justified by the low-Mach number limit whose description can be found for example in the monograph [9]. One key question is the smoothness of the solutions to (2) (and also (1)). Regularity properties of the Stokes system

$$\begin{aligned} \partial_t u - \operatorname{div} \mathbb{S} + \nabla \pi &= F \\ \operatorname{div} u &= 0 \end{aligned} \tag{3}$$

are critical to analyzing this question – see for instance [10] among many others. The main focus of this thesis is examining of the regularity properties of these systems under various conditions.

In [1] and [3], the focus is on the Stokes flow with a pressure-dependent viscosity tensor which may be non-linear in the variable Du. The stationary system of the form

$$
\begin{aligned}
-\operatorname{div}\mathbb{S}+\nabla\pi&=F\\
\operatorname{div}u&=0
\end{aligned}
\tag{4}
$$

is studied under the assumption that $\mathbb{S}=\mathbb{S}(\pi,Du)$.

In [1], which summarizes my doctoral thesis, I provide an estimate of the measure of the singular set, implying Hölder continuity of solution in two dimensions. The system is here endowed with the zero-Dirichlet boundary condition.

The second article on this topic [3] is just a small remark in which I use the method of Caffarelli and Peral [11] in order to locally improve the regularity of solution in means of Lebesque spaces.

The paper [2], coauthored with Jakub Tichý, investigates the non-linear Stokes problem, where the stress tensor is pressure-independent and is derived as the potential of an $N-$function:

$$\mathbb{S}(Du)=\partial_{ij}\Phi(|Du|)=\Phi'(Du)\frac{Du}{|Du|}$$

where Φ being in $C^1([0,\infty))\cap C^{1,1}((0,\infty))$ function, such that $\Phi'(s)$ is positive for $s>0$, non-decreasing, $\Phi'(0)=0$ and $\lim_{s\to\infty}\Phi'(s)=\infty$. This particular setting includes physically significant cases such as power-law fluids. Notably, we employed the full-slip boundary condition

$$u\cdot n=0,\qquad(\mathbb{S}(Du)n)\cdot\tau=0.$$

Using the Caffarelli and Peral approach mentioned earlier, we achieved higher integrability of solution.

The second article with the same coauthor [4] tackles the unsteady Stokes system (3) in two spatial dimensions equipped with the full-slip boundary condition. We dealt exclusively with the shear-thinning case, i.e.,

$$\partial^2_{|A|}\Phi(|A|):(B\otimes B)\sim(1+|A|^2)\frac{p-2}{2}|B|^2,\quad p\le 2$$

and we showed that the velocity gradient and the pressure possess the Hölder continuity for $p\ge 5/3$.

The regularity result provided in [2] is even stronger under the assumption that Φ has almost monotone second derivative and the problem is considered in three or more dimensions. There remained the question whether a similar improvement can be established in two dimensions.

This particular goal is addressed in [5] which was done in collaboration with Sebastian Schwarzacher. Besides that, we provide also results covering BMO and VMO regularity of solution. In order to reach this goal, we establish a new understanding of the boundary condition which is based on the reflection method mentioned in the upper paragraph. The paper is then concluded by a technique allowing the construction of counterexamples. In particular, we use the Riemann mapping theorem in order to show that the assumptions on the boundary smoothness considered in this article is sharp.

Finally, I turned my attention to compressible fluids, a popular topic at Institute of Mathematics of the Czech Academy of Sciences. Their flow is governed by (1) and may exhibit non-Newtonian behavior. This is the reason why this thesis contains [6] as it perfectly represents the combination of my interest in non-Newtonian fluids and my interest in compressible fluids. The last article of this thesis deals with the regularity of solutions which is obtained by a suitable analysis of the system linearized around the initial condition – it may come as no surprise that the system has a form

$$\begin{aligned} \varrho_0 \partial_t u - \mathcal{A}(Du_0)(Du) + p'(\varrho_0)\nabla\varrho &= F \\ \partial_t \varrho + \varrho_0 \operatorname{div} u &= G \end{aligned} \tag{5}$$

which is closely related to the Stokes system extensively studied in my earlier works – see [12]. In the scope of this article, we utilize Weis' theorem in order to proof the L^p–L^q regularity for (5) and this is later used together with the fix-point theorem in order to proof the existence of the strong solution to (1) for a short-time interval.

As a matter of fact, the field of the compressible non-Newtonian fluid is currently not satisfactory explored – here I would like to mention the overview in [13, Chapter 5] and works [14] and [15] besides others. The field itself deserves an intensive examination in the near future. This is the third reason for the presence of [6] in this thesis as it shows the intended direction of my future research.

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