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Different Concepts of Infinity in Mathematics

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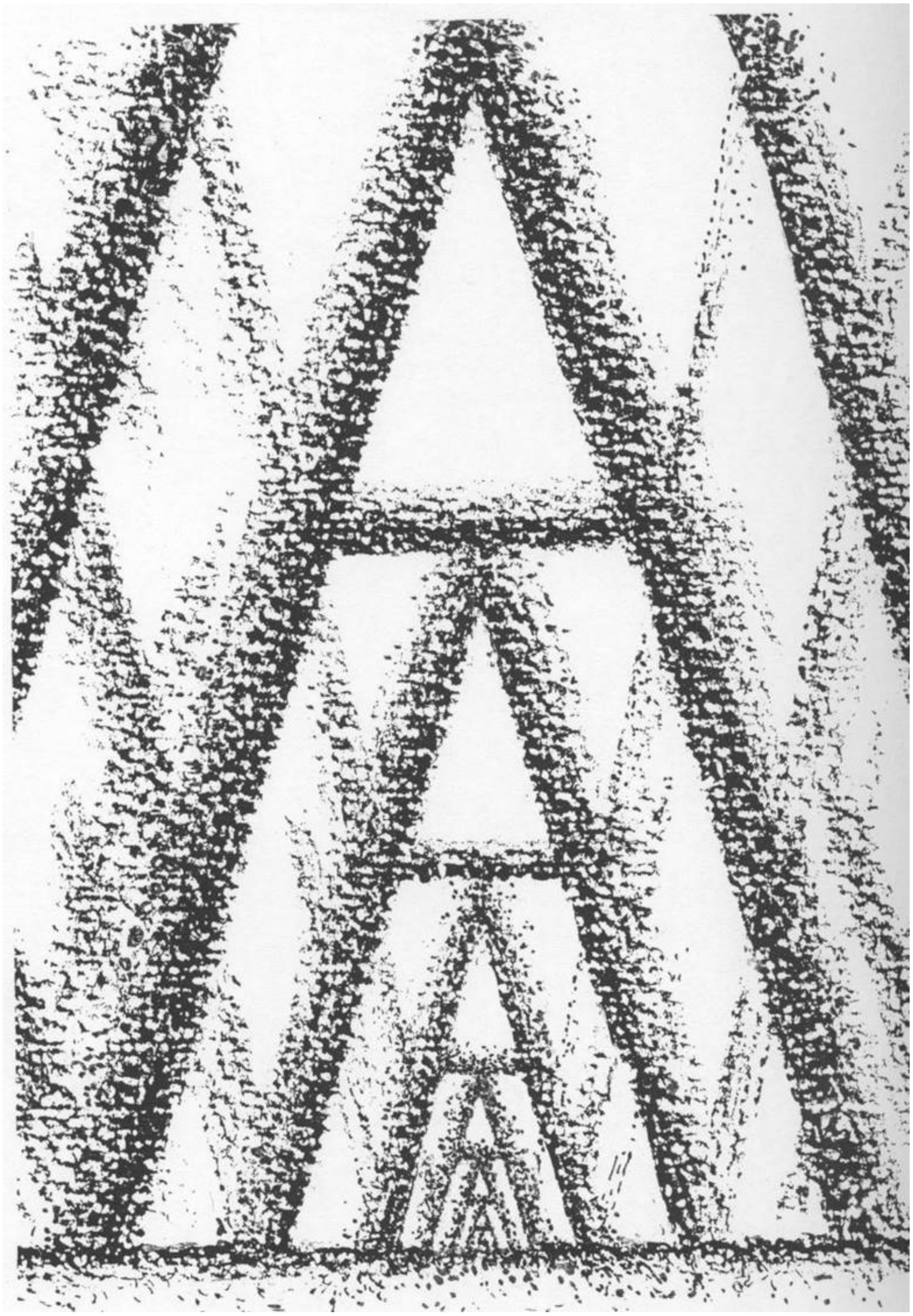

Věra Nováková: *Infinite A*

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Chapter 1

Introduction

> *Progress is made, not only in the arts, but in all other activities, not through the agency of those who are satisfied with things as they are, but through those who correct, and have the courage constantly to change anything which is not as it schould be.* (Isocrates: *Evagoras*).[1]

The question of infinity in mathematics seems to be solved and closed once and forever. Contemporary mathematics usually works with infinity in terms of sets both formally and conceptually. Formally, because set theory serves as a common language for all mathematical disciplines; conceptually, because mathematicians work with infinity in an axiomatic system based on Cantor's concept of actually infinite sets. The ordering of sets is expressed by ordinal numbers, and their sizes by cardinal numbers. The most widely used Zermelo-Fraenkel axiomatic set theory (ZF), as well as other axiomatic systems, are relatively consistent with respect to their axioms, although their absolute consistency cannot be proven according to Gődel's theorems.

The aim of this thesis is to compare different approaches to infinity. I follow up on Mancosu's question "Was Cantor's theory of infinite numbers inevitable?" (Mancosu 2009). In particular, I discuss Bolzano's concept of infinity and measurable numbers, Cantor's justification of his set theory and Vopěnka's Alternative Set theory.

The concept of infinity is closely related to the concept of real numbers. Bolzano's theory of measurable numbers from the early 1830s was developed forty years before Cantor's and Dedekind's theories of real numbers. Although there are some minor inconsistencies Bolzano proved almost all the properties of real numbers. We analyze whether this theory can indeed be considered a construction of real numbers.

Bolzano's other 1848 work, the *Paradoxes of the Infinite*, has been long considered an imperfect precursor to Cantor's theory. I show that Bolzano's theory of infinite quantities can be consistently interpreted in contemporary mathematics. The *Part-Whole Principle*, saying that "the whole is greater than the part", holds in contrast to Cantor's set theory.

Based on this Bolzano theory, I propose a notion of *size* for countable sets. While in Cantor's set theory all countable infinite sets have the same *cardinality*, namely $\aleph_0$, here the size of the part is less than the size of the whole. Of course, this size and the

[1] Bolzano's quotation on the title page of his first mathematical work *Consiserations on Some Objects of Elementary Geometry* from 1804. (Russ 2004, 27).

cardinality of finite sets coincide. The method is similar to *Numerosity Theory* (Benci & Di Nasso 2019), but in comparision it is constructive and the result does not depend on the choice of an ultarfilter. However, the sizes of the sets are only partially and not linearly ordered.

Cantor's set theory is well known, it needs no exposition here. What is not so well known, however, is its theological justification. I have described this peculiar history in (Trlifajová 2005). I am primarily concerned with Cantor's reasoning for the existence of infinite sets and his reconciliation with a whole tradition of more than two thousand years that did not admit the actual infinite. The paper was written in Czech and published in *Filosofický časopis* that I have translated into English.

The last topic I deal with is Petr Vopěnka's concept of infinity. His Alternative Set Theory (Vopěnka 1974) which originated almost fifty years ago is usually considered just one of the weaker versions of non-standard set theory. I believe, however, that the main contribution which has not yet been sufficiently appreciated, rests in its philosophical justification. It is based on phenomenology and offers a new interpretation of infinity.

Chapter 2

The problem of infinity

> *There is no actual (given complete) infinity. The Cantorians have forgotten this, and they have fallen into contradiction.* (Poincaré 1913, 484).
>
> *No one shall drive us out of the paradise which Cantor has created for us.* (Hilbert 1926, 191).

Philosophers have been concerned with the question of infinity since the Antiquity, actually since geometry went beyond direct evidence and introduced logical proofs. Early on it became apparent that a rational grasp of infinity often led to paradoxes and disputes. To develop a theory of mathematical infinity, two main problems need to be solved. The first concerns its existence, the second whether and how it can be measured.

2.1 The potential and the actual infinity

Aristotle formulated the first question of infinity, which became central to mathematics as well as to philosophy, and which has run through their history like a red thread. *Does infinity exist only in possibility or also in reality?* After careful analysis, Aristotle rejected the existence of infinity in reality: "It is plain from these arguments that there is no body which is actually infinite." (Aristotle Book III, Part 5, 206a) What remains, then, is that the infinite exists only in possibility.

Aristotle's question has been refined and generalized over time. Its form, with which medieval and modern thinkers came to terms, looked like this: *Does infinity exist only as potential or also as actual?* The *potential* infinity is one that may be ever-increasing or ever-decreasing but is never complete, whole, or finished. In contrast, the *actual* or *complete* infinity is conceived as a whole, is given *here and now* and can be treated as such. If we are dealing with infinite numbers, quantities or sets we are treating the actual infinity since *numbers*, *quantities* and *sets* are uniquely given, complete, and clearly determined. The notion of a *set* was invented just to grasp the actual infinite.

Scholastic philosophy adopted Aristotle's radical rejection of actual infinity and expressed it in the famous thesis of *infinitum actu non datur*. Most medieval and modern scholars adhered to it, and until the end of the 19th century, only potential infinity was considered acceptable. Moreover, the question of actual infinity has become a theological issue, because infinity was one of the attributes of God. As late as 1830 Gauss writes:

> But concerning your proof, I protest above all against the use of an infinite quantity as a *completed* one, which in mathematics is never allowed. The infinite is only *façon de parler*, in which one properly speaks of limits. (Dauben 1990, 120).

It was at the turn of the 19th and 20th centuries that the set concept, which works with the actual infinity, was adopted thanks to the German mathematician Georg Cantor, marking a radical turn in the development of mathematics. After some misunderstandings, rejections, and struggles, it was accepted by the mathematical community in the early 20th century. Almost all of mathematics was eventually rebuilt on a set basis that is commonly used until today.

2.2 The Part-Whole and the Hume Principles

However, even before that, some scholars have considered the actual infinity, notably Galileo, Leibniz, Arriaga and Bolzano. Then another question arises. *How to compare infinite collections?* We have to choose one of the following two principles:

1. The *Part-Whole principle* (PW): "The whole is greater than the part." It is the 5th Euclid Axiom.

2. *Hume's Principle* (HP):[1] "Two sets have the same size if and only if there is a one-to-one correspondence between them." It is also called *Cantor's Principle*.

The two principles cannot hold for infinite sets together. This inconsistency is well illustrated by Galileo's famous *Paradox of Reflexivity*. Galileo compares the infinite set of all natural numbers with the infinite set of their squares.

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots$$

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \ldots$$

On the one hand, Galileo explains, there are more numbers, including squares and non-squares, than the squares alone. On the other hand, there are as many squares as their roots because every square has its own root and every root its own square, and all numbers are roots. Galileo resolves this contradiction by denying the possibility of comparing infinite collections. (Galileo 1638/1914, 40–42).

2.3 The continuum

There is another fundamental phenomenon firmly connected to the infinity, and that is the continuum, the continuous extension. Mathematicians and philosophers have been aware of this connection from the beginning, and knew, or at least anticipated, that

[1] The origin of this term goes back to Frege who states this principle with reference to Hume's *Treatise of Human Nature* Book 1, Part 3, Section 1 in *Grundlagen der Arithmetik*, §63.

once they formed one, it would necessarily be reflected in the other. The continuum and the infinite are two opposing sides of a coin that are interconnected but far from simply.[2]

The basic questions concerning continuum are analoguous to those of infinity. Aristotle's conception is that a continuum is potentially infinitely divisible into ever smaller contiuous parts. However, the school of atomism represented by Democritus held that continua are composed of infinitely many discrete particles.

The second question, *how to compare continua*, encounters the same conflict between the *Part-Whole Principle* and *Hume's Principle* as Leibniz had already demonstrated in (Pacidius Philalethi, Opus. 611). Points on the diagonal of a rectangle could be put into one-to-one correspondence with the points on one side, and these with the points on a part of the diagonal equal in length to the side, so that there would be as many points in the part of the diagonal as in the whole. Leibniz refused to compare infinite collections in size, although he did not reject actual infinity *per se*.[3]

Traditional examples of continuum are space, time and motion. These examples as well as many others can be expressed in a numerical system as a ratio to a chosen unit of the same kind. The basic type of continuum is a straight line. However, some line segments are not in a rational ratio to the unit line. The problem was how to express all the lengths arithmetically, the *arithmetization of continuum*.

2.4 Mathematical analysis

One of the most important inventions of modern mathematics was undoubtedly the infnitesimal calculus at the turn of the 17th and 18th centuries. However, because at that time the *arithmetization of the continuum* had not yet been developed, there was no theory of real numbers or precise rules for infinitesimal numbers, ambiguities and errors occurred. Moreover, infinitesimal numbers were attacked by some philosophers, such as Berkeley's sarcastic criticism in *The Analyst*.

In principle, there are two separate tracks for development of mathematical analysis (Bair & al. 2013, 889).

(A) the Weierstrassian approach (in the context of an *Archimedean* continuum);

(B) the approach with indivisibles or infinitesimals (in the context of what is referred to as a *Bernoullian* continuum.)

The (A) track actually assumes the Aristotelian continuum where the *Archimedean axiom* applies. For any two quantities a,b

$$(\forall a)(\forall b)(\exists n \in \mathbb{N})(n \cdot a > b).$$

[2] According to Leibniz "there are two labyrinths of the human mind, one concerning the composition of the continuum, and the other concerning the nature of freedom, and they arise from the same source, infinity." (Look 2022).

[3] Both Bolzano and Cantor quote with satisfaction the passage from Leibniz's letter "I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author." (Cantor 1883, 78), (Russ 2004, 593.)

The second (B) track, assuming non-Archimedean system encompassing infinitesimals, had been used from the beginning of the invention of the infinitesimal calculus by Leibniz, Bernoulli, Euler and others until the end 19th century.

At the end of the 19th century (A) track prevailed and is commonly used until today. It is based on the Weierstrassian $\varepsilon - \delta$ technique and Cantor's and Dedekind's constructions of real numbers. The infinitesimal approach (B) was considered "at best, a kind of evolutionary dead-end or, at worst, altogether inconsistent." (Bair & al. 2013, 891).

The (B) track was not rehabiltated until the 1960s when Abraham Robinson created *Non-Standard Analysis* using ultrafilters.

> The concepts and methods of contemporary Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers. (Robinson 1966: p. vii)

Infinitely small and infinitely large quantities of *Non-Standard Analysis* correspond to each other, they are mutually inverse. In the sense that Blaise Pascal once argued in *De l'ésprit géometrique*:

> Ces deux infinis, quoique infinement différents, sont néanmoins relatifs l'un à l'autre, de telle sorts que la connaissance de l 'un mène nécessairement a la connaissence de l'autre. [4] (Pascal 1866, 295).

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[4] These two infinities, though infinitely different, are nevertheless relative to each other, in such a way that knowledge of one necessarily leads to knowledge of the other.

Chapter 3

Bernard Bolzano

The possible is what stands in contradiction with no pure conceptual truth.
(Bolzano 1851, 610).

Bernard Bolzano (1781–1848) was an eminent Bohemian scholar, who dealt with many fields, mathematics, logic, philosophy, theology, and even aesthetics. He was a Catholic priest and a professor in the Philosophical Faculty of the Charles University in Prague. For his liberal intellectuality and progressive theological and political ideas he was excluded from the University in 1820 and was forbidden to teach, preach, or publish and had to sustained on a meagre pension "graciously granted" by the Austrian emperor. However, he had time he needed to elaborate and write his new foundation of logic and philosophy. This was published in a three-volume book under the title *Theory of Science (Wissenschaftlehre)* (WL) in 1837. After that, he devoted a great effort to elaborate a new foundation of mathematics under the title *Pure Theory of Numbers (Reine Zahlenlehre)* (RZ). Bolzano's program was to built a solid analytical ground for the whole of mathematics, as he already outlined in *Contributions to a Better-Grounded Presentation of Mathematics* of 1810. The realization of this project was considerably developed but not completed. Before his death he happened to write a famous book *Paradoxes of the Infinite (Paradoxien des Unendlichen)* (PU) published posthumously in 1851. All these works are translated into English in (Russ 2004).

3.1 Measurable numbers

Bolzano's *Reine Zahlenlehre* from the early 1830s is a huge Bolzano's project whose goal is to describe the theory of all numbers by the purely analytic method. Its most interesting and most discussed part is the VII$^{\text{th}}$ section.

The earlier Bolzano's tiny book *A Purely Analytic Proof that between any two Values which give Results of Opposite Sign, there lies at least one real Root of the equation (Reine Analytischer Beweis)* (Bolzano 1817) is a diamond among historical mathematical works. It includes a definition of the continuity of a function, states the Cauchy (or rather Bolzano-Cauchy) criterion, contains proofs of the Cauchy (Bolzano-Cauchy) theorem, the supremum theorem, and the intermediate value theorem which gave the book its title.[1] All theorems and their proofs are analytic, which means they

[1]There has even been some discussion as to whether Cauchy may have plagiarized parts of this book

are formally proved from basic definitions, without reference to geometry or applied mathematics. It is well known that the proof of the key Bolzano-Cauchy theorem, which states that any Bolzano-Cauchy sequence has a limit, is incomplete. The reason is quite obvious. There was no theory of real numbers at the time, so it was impossible to describe a number that the sequence converges to. Bolzano could not prove the existence of a limit, he actually only proved its uniqueness.

Bolzano was apparently aware of the missing theory of real numbers and wanted to repair it in the VII$^{\text{th}}$ Section of *Reine Zahlenlehre*. It contains an original construction of *measurable numbers* which Bolzano proved to have all the properties of contemporary real numbers. The problem is that there occur some minor inconsistencies. Bolzano's intuition and deep insight seem to have preceded his ability to formulate the theory of measurable numbers with complete precision. Moreover, he had never finished this work. It was preserved as a manuscript not ready for publication in Vienna's State Library until the second half of the last century when Karel Rychlík published it in (Rychlík 1962). The second Berg's critical edition including important supplements was published later. (Bolzano 1976).

The great debate as to whether this is indeed a construction of real numbers has not yet been convincingly concluded. An affirmative answer would mean that Bolzano's construction predates Cantor's and Dedekind's constructions by about forty years. Scholars' opinions vary; ranging from condemning the theory as inconsistent (van Rootselaar 1966), to classifying it as a precursor to Cantor's theory of real numbers (Rychlík 1962), (Šebestík 1992), (Rusnock 2000) to believing that one small change in the initial definition can completely correct the entire theory, (Laugwitz 1966), (Russ & Trlifajová 2016).

3.1.1 Infinite number expressions

Bolzano starts with *infinite number expressions* which are formed by using an infinite number of arithmetic operations. He presents several examples

$$A = 1+2+3+4+\ldots \text{in inf.}$$

$$B = \frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots \text{in inf.}$$

$$C = (1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{8})(1-\frac{1}{16})\ldots \text{in inf.}$$

$$D = a+\frac{b}{1+1+1+\ldots \text{in inf.}}$$

where a,b are a pair of integers.

Bolzano selects some of the expressions such that for each natural number q they are elements of a half-open interval $[\frac{p}{q},\frac{p+1}{q})$ for some integer p. He calls these expressions *measurable numbers*, C and D for instance. *Infinitely small* positive numbers lie in the intervals $[0,\frac{1}{n})$ for every natural number n, C is the example. *Infinitely great* positive numbers are greater than any n, for instance A.

in his 1821 *Cours d'Analyse* (Grattan-Guinness 1970). According to (Bair et al. 2020), however, Cauchy and Bolzano stated the same (Bolzano-Cauchy) criterion for convergence of sequences at similar times, but independently.

The easiest way is to interpret Bolzano's infinite expressions as sequences of partial results. We get sequences of rational numbers. Measurable numbers correspond to Bolzano-Cauchy sequences, infinitely small numbers to sequences converging to 0 and infinitely great numbers to divergent sequences.

The last step of the construction is the introduction of *equality* and *order*. Two measurable numbers X, Y are *equal* if their difference $|X - Y|$ is infinitely small. X is *greater* than Y if their difference $X - Y$ is positive and not infinitely small. All infinitely small numbers are thus equal to 0 and they are closed on addition and multiplication.

Bolzano actually factorizes measurable numbers modulo equality, however, he does not distinguish between the "old" measurable numbers and their factor-class, the "new" measurable numbers. This is a correct, albeit unusual, way; however, it has been the problem for some interpreters of Bolzano's work. Now, Bolzano states and proves all properties of real numbers: his "new" measurable numbers form a linearly ordered, dense, complete, Archimedean field. He can finally prove the Bolzano-Cauchy Theorem and the Supremum Theorem, which he formulated already in 1817.

3.1.2 The standard interpretation

The basic notions of Bolzano's theory of measurable numbers and Cantor's theory of real numbers are summarized in this table.

Bolzano's theory	**Cantor's theory**
infinite number expressions	sequences of rational numbers
infinitely small numbers	sequences converging to zero
infinitely large numbers	divergent sequences
measurable numbers	Bolzano-Cauchy sequences
the equality of measurable numbers	the difference converges to zero
measurable numbers with the equality	real numbers

3.1.3 A non-standard interpretation

Besides the well-known Cantor's and Dedekind's construction of real numbers, there is a construction using the non-standard model of rational numbers (Albeverio & al. 1986). We start with the sequences of rational numbers $\mathbb{Q}^{\mathbb{N}}$ and a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{N}$. The ultraproduct $\mathbb{Q}^* = \mathbb{Q}^{\mathbb{N}}/\mathscr{U}$ is a linearly ordered non-Archimedean field. Infinitely small numbers $\mathbb{Q}_i$ are less than any rational number. Infinitely large numbers are greater than all natural numbers defined as usually. Finite non-standard numbers $\mathbb{Q}_f$ are those that are not infinitely great. Two finite elements $x, y \in \mathbb{Q}_f$ are equivalent, $x \doteq y$, if their difference $|x - y|$ is infinitely small. By factorisation we obtain the complete linearly ordered Archimedean field of real numbers, $\mathbb{R} = \mathbb{Q}_f / \doteq$. The method is similar to Robinson's non-standard analysis, however, the goal is different.

We can also interpret Bolzano's measurable numbers in this way. The advantage is that infinitely small numbers do not disappear. Measurable numbers correspond to non-standard finite rational numbers $\mathbb{Q}_f$. Both form a commutative non-Archimedean ring. Bolzano's equality and the non-standard equivalence on this ring are defined by the same way. Thus, the result is identical, the real numbers.

Bolzano's theory	Non-standard theory
infinite number expressions	sequences of rational numbers $\mathbb{Q}^{\mathbb{N}}$
measurable and infinitely large numbers	non-standard rational numbers $\mathbb{Q}^*$
infinitely small numbers	infinitely small numbers $\mathbb{Q}_i$
measurable numbers	finite non-standard rational numbers $\mathbb{Q}_f$
the equality of measurable numbers	the equivalence relation $\doteq$
measurable numbers with the equality	real numbers $\mathbb{R} = \mathbb{Q}_f / \doteq$

3.1.4 The mathematical analysis

An important issue of the time was to find the clear foundations for the infinitesimal calculus. Bolzano was at a crossroad whether to use Weierstassian (A) track and a type of ε-δ method or (B) track and infinitesimal calculus. The introduction of infinitely small numbers in the VII$^{\text{th}}$ Section of *Reine Zahlenlehre* was a challenge. He actually obtained a rich non-Archimedean continuum encompassing infinitely small quantities, B-track.

> It is a most interesting historical fact that Bolzano, the best logical mind of the generation, made a real effort to clear the matters. He was possibly the only one to see the problems related the difference between the two continua: the rich Leibnizian continuum and, as he called it, its 'measurable' subset - the set of Weierstrassian real numbers. Bolzano makes it very clear that the field of 'measurable numbers' constitutes only an Archimedean subset of a continuum enriched by non-measurable - infinitely small and infinitely large quantities. ...No doubt, since Robinson has shed new light on the latter, historians will approach the Bolzano manuscript with new eyes and the relation between Bolzano's measurable numbers and non-measurable quantities and Robinson's standard and non-standard numbers will be clarified. (Lakatos 1978, 53).

In his following work *Theory of Functions* (Russ 2004), also preserved only as a manuscript, Bolzano decided for A-track. He works with his new measurable numbers and does not mention infinite quantities at all. He defines key notions of analysis in a similar way as the later ε-δ calculus.[2]

Bolzano was clever enough to see that his infinitely small numbers as well as his infinite number expressions are not convenient for the infinitesimal calculus. We interpret them as non-standard *rational* numbers $\mathbb{Q}^*$. The important *transfer principle* does not apply between $\mathbb{R}$ and $\mathbb{Q}^*$. Functions defined on the real numbers cannot be easily extended on non-standard rational numbers.

[2] For instance, the continuity of a function in a point x:

> Therefore supposing the value Fx is measurable, as well as the value $F(x+\Delta x)$(at least, for the latter starting from a certain value of the difference Δx and for all smaller values bellow), but the difference $F(x+\Delta x) - Fx$, in its absolute value, becomes and remains smaller than any given fraction $\frac{1}{N}$, providing only that Δx is taken small enough then I say that the function Fx varies continuously for the value x. (Russ 2004, 448).

3.1.5 Recent papers

After the publication of our paper (Russ & Trlifajová 2016), three new works on this topic appeared. Fuentes-Guillén (2021) in *Bolzano's Theory of messbare Zahlen: Insights and Uncertainties Regarding the Number Continuum*, investigates particularly the conceptual frameworks of Bolzano's theory and his insights and uncertainties with regard to the notions of number and quantity. He concludes that Bolzano remained still bound to a not-yet-modern conception of mathematics and numbers.

Anna Bellomo (2022) in her dissertation *Sums, Numbers and Infinity: Collections in Bolzano's Mathematics and Philosophy* carefully analyses Bolzano's proofs, asks in what sense it is the arithmetization of continuum and challenges the usual sequence interpretation to "open up the opportunity of truly understanding and appreciating Bolzano's measurable numbers as part of his broader attitudes.". Nevertheless, she concludes that Bolzano's measurable numbers are an attempt at a rigorous treatment of real numbers (Bellomo 2022, 98).

Quite recently, another dissertation on Bolzano was published. Marlena Fila's *Number Structures in the Mathematical Writings of Bernard Bolzano* (2023) investigates two Bolzano's books *A purely Analytic Proof* and the VII$^{\text{th}}$ Section of *Reine Zahlelehre*. She uses the above mentioned model to show that measurable numbers correspond to the construction of real numbers from non-standard rational numbers.

All these scholars are right, of course. Fuentes-Guillén and Bellomo are trying to faithfully locate Bolzano's place in the history of mathematics. My intention is somewhat different. I want to show that Bolzano's theory can be interpreted consistently and can serve as a dignified alternative to Cantor's theory. What is interesting is that all these constructions, Bolzano's measurable numbers, Cantor's and Dedekind's real numbers, constructions via non-standard rational or real numbers, lead to the same theoriy of real numbers, a complete, dense, linearly ordered, non-Archimedean field.

3.2 Infinite quantities

Bolzano's *Paradoxes of the Infinite* (PU) (Russ 2004) is the key book to the comprehension of his concept of infinity. It was written in 1848, just before Bolzano died, and published afterwards in 1851. Bolzano's aim is to demonstrate that the right calculation with the infinite does not bring any paradoxes. Therefore, he first clarifies his concepts associated with infinity. He starts with "*the general theory of quantity* where such paradoxes are not missing even in number theory." (PU §29).

3.2.1 The actual infinity

Bolzano thoroughly explains the basic notions of his theory. (PU §§3 - 9). Nearly the same explanations can be found in the *Theory of Science*. (WL §§82 - 86.) A *collection* of certain things is a "whole consisiting of certain parts which are connected by the conjunction *and*". It can be given as a certain list, for instance "the sun, the earth and the moon, or the rose and the concept of a rose". or can be denoted by some idea A, which we call the collection of all A. A *multitude [Menge]* is a collection that is conceived such that the arrangement of its parts is unimportant. Bolzano introduces multitudes for the same reason as Cantor introduces sets: in order to capture an infinite plurality altogether, it must be collected in a single collection, in a whole.

The corner stone of Bolzano's logic was the concept of *truth in itself* or *objective truth*. It is any proposition which states something as it is, notwithstanding whether this proposition has been thought or spoken by anybody. (WL §25).

Bolzano proved the existence of *the infinite multitude of truths in themselves* First, he proved that "there is at least one objective truth". This is true since the opposite claim is self-contradictory (WL §32). If there is a true proposition A, then the proposition B: "The proposition A is true" is also true and it is different from A. The next proposition C: "The proposition B is true" is also true, and so on. We eventually have and infinite multitude of truths in themselves.

This multitude of truths is actually infinite for we deal with objective truths and it does not depend whether somebody knows or claims them. (Šebestík 1992, pp. 446-447). Nevertheless, Bolzano supported the existence of this multitude by the theological argument:

> Thus we must attribute to Him [God] a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths. (PU §11.)

3.2.2 The Part-Whole Principle

Bolzano was aware of the existence of a one-to-one correspondence between some infinite multitudes. However, he warned against the claim that a one-to-one correspondence would imply the equality of multitudes and insisted on preserving the *Part-Whole Principle*.

> Merely from this circumstance [the existence of an one-to-one principle] we can in no way conclude *that these multitudes are equal to one another if they are infinite* with respect to the plurality of their parts (i.e. if we disregard all differences between them). But rather they are able to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. (PU §21).

He adds that only in some cases we can make such conclusions.

> An equality of these multiplicities can only be concluded if some other reason is added, such as that both multitudes have exactly the same *determining ground*, e.g. they have exactly the same *way of being formed*. (§21).

To determine an object means to describe all representations that the objects falls under. The determination is complete if the representation of an object is unique. Thus, a point, the area and the diameter completely determine a circle, because all of its properties can be uniquely determined. (Šebestík 1992, 460).

3.2.3 Infinite series

While in *Reine Zahlelehre* Bolzano deals with infinite numbers expressions that are measurable, i.e. that can be interpreted as Bolzano-Cauchy sequences of rational numbers, in *Paradoxes of the Infinite*, he particularly deals with infinitely great number expressions which can be interpreted as divergent series. They are formed only by addition, so they are infinite series.

Bolzano starts with infinite series of natural numbers, but later he takes in account as well infinite series of other quantities. He introduces examples in §29 and §33.

$$N = 1+1+1+1+\dots+\text{in inf.}$$

$$N_n = \dots 1+1+1+\dots+\text{in inf.}$$

$$P = 1+2+3+4+\dots+\text{in inf.}$$

$$S = 1+4+9+16+\dots+\text{in inf.}$$

The series N represents the multitude of all natural numbers. The series N_n is similar, but the first n terms are omitted, it represents the multitude of natural numbers greater than n.

While in the usual mathematical understanding all these divergent series have the same infinite sum, for Bolzano they are examples of various infinite quantities. Several important principles for adding, subtracting, and comparing them follow from the way Bolzano treats and comments on them. (Trlifajová 2024).

3.2.4 The sequence interpretation

We interpret Bolzano's infinite series as sequences of partial sums to demonstrate they form a consistent system in accordance with Bolzano's principles (Trlifajová 2018). By converting the series to sequences, the requirement of "one and the same multitude" of terms is satisfied. Each term of a sequence has its specific uniquely determined place. The omitted terms of series, as in the case of N_n, are simply replaced by zero, so specific differences, as between N and N_n, are naturally preserved.

- $N = 1+1+1+1+\dots$ in inf. $\sim (1,2,3,\dots) = (n)_n$.
- $N_n = \underbrace{0+\dots+0}_{n}+1+1+\dots$ in inf. $\sim (\underbrace{0,\dots,0}_{n},1,2,3,\dots)$.
- $P = 1+2+3+4+\dots$ in inf. $\sim (1,3,6,10,\dots) = (\frac{n\cdot(n+1)}{2})_n$.
- $S = 1+4+9+16+\dots$ in inf. $\sim (1,5,16,32,\dots) = (\frac{n(n+1)(2n+1)}{6})_n$.
- $n = \underbrace{1+\dots+1}_{n}+0+0+\dots$ in inf. $\sim (1,2,3,\dots,n,n,n,\dots)$ for $n \in \mathbb{N}$.

The sum and the product of two sequences is defined componentwise. According to Bolzano, the value of a series undergoes no change when we change the order of finitely many terms. Therefore we define that the equality and the order of sequences is valid if it is valid from a sufficiently great index. In fact, it means that the equality and the order is valid modulo the Fréchet filter. One sequence is infinitely greater than another one if the former is greater than any finite multiple of the latter one.

In this interpretation, Bolzano's series of natural numbers are uniquely interpreted as non-decreasing sequences of natural numbers. They form a *partial ordered non-Archimedean commutative semiring.* Bolzano's assertions on infinite quantities presented in *Paradoxes of the Infinite* are valid. This is a consistent and meaningful structure and we shall employ it in the next section for the investigation of sizes of countable sets.

Anna Bellomo and Guillaume Massas (2021) developped a novel interpretation of this part of Bolzano's theory. They assume that the product operation of his infinite quantities is not in general commutative. Although the authors present a slightly different interpretation based on an iterative ultraproduct they also show that Bolzano's theory of infinite quantities can be consistently interpreted.

3.2.5 The extended domain

In the following part of *Paradoxes of the Infinite*, Bolzano extends the domain of infinite series for more quantities. He does not elaborate it too much, there are only some remarks, but apparently his intention was to create a general theory of quantities. He gives further examples. If N is an infinitely great quantity, then $1/N$ represents an infinitely small quantity, $N^2 = N + N + N + \dots$ in inf. is a quantity infinitely greater than N. If α, β are quantities then also $\alpha \cdot N = \alpha + \alpha + \alpha + \dots$ in inf., $\beta \cdot N = \beta + \beta + \beta + \dots$ in inf. are quantities. The relation $(\alpha \cdot N) : (\beta \cdot N)$ equals to $\alpha : \beta$.

Also this extension can be consistently interpreted. When we consider α, β to be Bolzano's measurable numbers, i.e. our real numbers according to the previous section, then all these quantities can be interpreted as *sequences of real numbers.* We define the same structure on the set of sequences of real numbers as in the previous case. We get a *partial ordered non-Archimedean commutative ring*. It contains finite, infinitely small and infinitely large quantities which have arbitrary finite, infinitely small and infinitely great ratio one to another. The question arises whether this structure could serve as a basis for calculus infinitesimals. It also could be the reason for Bolzano to develop this theory.

This was attempted by Terence Tao, who described the so-called "cheap version of non-standard analysis" defined on this very structure (Tao 2012). This version is constructive because it does not require any kind of the *Axiom of Choice*. The infinitesimal calculus can be partially reconstructed in it. Obviously, it is less powerful than the full version. The structure of sequences of real numbers factorized by the Fréchet filter is not an integral domain, it contains nontrivial zero divisors such as $(1, 0, 1, 0, \dots)$ and other useless elements.

The only way to avoid this is to use a non-principal ultrafilter in the definition of equality and ordering of sequences instead of the Fréchet filter. Then we obtain an ultraproduct that is a *linearly ordered non-Archimedean field* where the transfer principle holds and where the mathematical analysis can be based on infinitely small numbers.

Ultrafilters and their wonderful properties are the result of modern logic of the 20th century. Axiom of choice is needed to prove their existence. Bolzano, of course, could not have known that. Jan Berg aptly named his book on Bolzano *Ontology without ultrafilters* (Berg 1992).

3.3 Set sizes

The above described Bolzano's structure of infinite series of natural numbers can be used to express sizes of some countable sets, so that the Part-Whole Principle is preserved: the sizes of natural numbers, rational numbers, and of their subsets, unions and Cartesian products. (Trlifajová 2024). The method is similar to that of *Numerosity Theory* but in comparison, it is constructive and set sizes are uniquely determined. However, set sizes are only partially, not linearly, ordered. *Quid pro quo.*

If we count a large or even an infinite multitude of things from a set A then it is often appropriate to divide these things according to some rule into smaller finite subsets $A_n \subseteq A$ such that $A = \bigcup\{A_n, n \in \mathbb{N}\}$.[3] Subsets $A_n \subseteq A$ are called *components*. The size of A is then equal to the sum of finite cardinalities of its components.

$$|A_1| + |A_2| + |A_3| \dots \text{in inf.}$$

This is Bolzano's series which we interpreted as a sequence of partial sums, we call it a *size sequence* of the set A.

$$\sigma(A) = (|A_1|, |A_1| + |A_2|, |A_1| + |A_2| + |A_3|, \dots)$$

3.3.1 The canonical arrangement

Obviously, there are infinitely, even uncountably, many arrangements of the set A into components. In this point, I refer to Bolzano's claim that the same size of two multitudes between which there is a one-to-one correspondence "can only be concluded if some other reason is added, such as that both multitudes have exactly the same *determining ground*". Therefore, we define a canonical arrangement of the set A to depend on its *determining ground*.

We start with the natural numbers $\mathbb{N}$. The n-th component of its canonical arrangement contains just the number n, it is $\{n\}$. The size sequence of natural number is denoted as α.

$$\sigma(\mathbb{N}) = (1, 2, 3, \dots) = \alpha.$$

This arrangement of natural numbers forms a kind of skeleton which is gradually extended according to the following rules.

1. Subsets of canonical arranged sets should preserve the same arrangement, it means that if B is a subset of a canonically arranged set A then its n-th component $B_n = A_n \cap B$.

2. The Cartesian product $A \times B$ of two canonically arranged sets A, B is also canonically arranged. Its n-the component $(A \times B)_n$ is the union of $A_i \times B_j$ where $n = \max\{i, j\}$, the "n-th frame".

3.3.2 Size sequences

Size sequences are non-decreasing sequences of natural numbers which we consider as the interpretation of Bolzano's series. Their sum and product are defined componentwise, their equality $=_{\mathscr{F}}$ and ordering $<_{\mathscr{F}}$ modulo Fréchet filter. This implies, among

[3] We consider the natural numbers $\mathbb{N}$ as positive integers, $\mathbb{N} = \{1, 2, 3, \dots\}$

other things, that size sequences of all sets with the same finite cardinality have an equal size.

Theorem 1. Let A, B be two canonically arranged sets. The size of their union is the sum of their sizes minus the size of their intersection.

$$\sigma(A \cup B) = \sigma(A) + \sigma(B) - \sigma(A \cap B)$$

The size of their Cartesian product is the product of their sizes.

$$\sigma(A \times B) = \sigma(A) \cdot \sigma(B).$$

The size of a finite sets is equal to its cardinality.

$$A \text{ is finite } \Rightarrow \sigma(A) =_{\mathscr{F}} |A|.$$

The Part-Whole Principle is valid. If A is a proper subset of B

$$A \subset B \Rightarrow \sigma(A) <_{\mathscr{F}} \sigma(B).$$

The structure $(\mathbb{N}^{\mathbb{N}}, +, \cdot, =_{\mathscr{F}}, <_{\mathscr{F}}, <<_{\mathscr{F}})$ is a partial ordered non-Archimedean semiring. Nevertheless, it is not linearly ordered. For instance size sequences of odd numbers is $\sigma(\mathbb{O}) = (1, 1, 2, 2, 3, \dots)$ and even numbers is $\sigma(\mathbb{E}) = (0, 1, 1, 2, 2, \dots)$. Surely,

$$\sigma(\mathbb{E}) \leq \sigma(\mathbb{O}) \leq \sigma(\mathbb{E}) + 1 \quad \text{and} \quad \sigma(\mathbb{E}) + \sigma(\mathbb{O}) = \alpha.$$

We define that *a is infinitely smaller than b* if every finite multiple of a is smaller than b, $a <<_{\mathscr{F}} b$, and *a is equal in order to b* if neither a is infinitely smaller than b nor b is infinitely smaller than a, $a \approx_{\mathscr{F}} b$.

3.3.3 Natural and rational numbers

Then we can prove the relationships between important subsets of natural numbers.

Theorem 2. For $p, k \in \mathbb{N}$ let $\mathbb{S}_p$ be the set of all p-th powers of natural numbers, $\mathbb{P}$ the set of all primes and $\mathbb{M}_k$ the set of all k-multiples of natural numbers. Then the size of all powers is infinitely smaller than the size of all primes that is infinitely smaller than the size of all k-multiples.

$$\sigma(\mathbb{S}_p) <<_{\mathscr{F}} \sigma(\mathbb{P}) <<_{\mathscr{F}} \sigma(\mathbb{M}_k) \approx_{\mathscr{F}} \alpha$$

The unit interval of rational numbers $\mathbb{I} = (0, 1]_{\mathbb{Q}}$ is represented as a subset of $\mathbb{N} \times \mathbb{N}$ of coprime pairs natural numbers (k, m) such that $k \leq m$. The size sequence of the unit interval $\sigma(\mathbb{I})$ is equal to $(\Phi(n))_n$ where $\Phi(n)$ is the *totient summatory function*.[4]

It turns out that if we represented positive rational numbers as a subset of $\mathbb{N} \times \mathbb{N}$ of coprime pairs of natural numbers then the interval of rational numbers greater than 1 would have the same size as the unit interval $\mathbb{I}$. It is not satisfactory. Much better seems to represent rational numbers as mixed fractions, i.e. as the Cartesian product of integers and the unit interval $\mathbb{Q} = \mathbb{Z} \times \mathbb{I}$. This representation of rational numbers is *homogeneous* which means that all intervals of rational numbers of the same length have the same size.

[4]The totient summatory function $\Phi(n) = \sum_{i=1}^{n} \varphi(i)$ where the Euler function $\varphi(n)$ is the number of coprime numbers less than n.

Theorem 3. The size of all integers $\mathbb{Z}$

$$\sigma(\mathbb{Z}) = 2\alpha + 1.$$

The size $\sigma(\mathbb{I})$ of the unit interval $\mathbb{I} = [0,1)_{\mathbb{Q}}$ of rational numbers

$$\sigma(\mathbb{I}) = (\Phi(n))_{n\in\mathbb{N}} \approx_{\mathscr{F}} \alpha^2,$$

The size of rational numbers

$$\sigma(\mathbb{Q}) = 2\varphi \cdot (\alpha + 1) - 1 \approx_{\mathscr{F}} \alpha^3.$$

This result may seem surprising at first glance. However, the rational numbers are infinite not only "in length" as integers but also "in depth". The nature of this latter infinity is different from the former. The interval of rational numbers between any two successive integers is literally infinitely denser than the set of integers; no wonder its size is infinitely greater than α. The rational numbers are the union of infinitely many such intervals.

Chapter 4

Georg Cantor

> *All things, whether finite or infinite, are definite and, with the exception of God, can be determined by the intellect.* (Cantor 1883, 76).

Cantor's set theory is well known via the Zemelo-Fraenkel axiomatic system and needs no exposition. What is not so well known is its justification and the circumstances of its admission. This peculiar history give reasons for philosophical doubts about the necessity if not the adequacy of Cantor's sets.

4.1 Real numbers

At the beginning of the 1870s, a young, talented German mathematician Georg Cantor investigated the problem of the uniqueness of trigonometric series. In doing so, he realised that a correct solution required precise definitions of real numbers, which at that time had not yet been established.

In his 1872 paper, he constructs real numbers using the so-called *fundamental sequences*, now *Bolzano-Cauchy sequences*, of rational numbers. He defined that two sequences are equal if their difference is a sequence converging to zero. Cantor actually factorized fundamental sequences modulo equality and thus obtained the structure of real numbers.

Cantor identifies the points of a line and the real numbers. On the one hand, for each point we find a fundamental sequence converging to it. On the other hand, the assignment of just one point to any fundamental sequence is not self-evident, and Cantor admits that this identification must be provided by an axiom. The *arithmetization of continuum* was thus completed.

In next papers, Cantor shows that algebraic numbers are countable, i.e. they can be numbered by natural numbers or there is a one-to-one correspondence between them and natural numbers. He also proves that real numbers in any interval are not countable, there are more of them. Based on this result, Cantor introduces the notion of *cardinality* by means of a one-to-one correspondence. Later, Cantor himself was astonished when it turned out that the cardinality of an arbitrary n-dimensional bounded space is equal to the cardinality of one-dimensional line segment: "I see it but I don't believe it." (Dauben 1990, 55).

4.2 The actual infinity

Cantor eventually published *Foundations of a General Theory of Manifolds*, subtitled *A Mathematical-Philosophical Study in the Theory of the Infinite* (Cantor 1883) containing the foundations of the set theory based on two main principles.

1. The existence of actually infinite sets.
2. *Hume's Principle*: Two sets have the same size if there is a one-to-one correspondence between them.

While the second principle was vindicated by his own research, the first principle was in contrary to the longstanding conviction that the only infinity we can treat is potential. Cantor felt obliged to defend his position. For him "the *essence* of mathematics lies in its *freedom.*".

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. (Cantor 1883, 79).

However later, Cantor did not consider this justification sufficient and tried to find a deeper one, more in accordance with the philosophical and theological tradition.

4.3 Theological reasoning

He received unexpected support from the Roman Catholic Church. In 1879, the newly elected Pope Leo XIII issued one of his first encyclicals, *Aeterni Patris*. He sought to revive the tradition of philosophical thinking, especially Thomistic philosophy. The important theologian, Konstantin Gutberlet drew upon Cantor's theory in a defense of his own views on infinity.

Cantor started to study theology and pointed and give reasons why many philosphers including St. Augustine and St. Thomas Aquinas rejected actual infinity. He was persuaded that it was on the basis of *protos peudos* that infinite numbers would have to have the same properties as finite numbers. There was only one answer. To introduce new infinite ordinal and cardinal numbers, and so prove that they can exist.

Cantor claims that the natural numbers, both together and separately, as well as all infinite numbers, ordinal and cardinal, exist in a higher degree of reality from eternity as thoughts in God's intellect.

> All these paricular modes of the transfinite have existed from eternity as ideas in Divine intellect. (Hallett 1984, 21).

Cantor distinguishes three main forms of manifesting the actual infinite in *On the Various Standpoints With Regard to the Actual Infinite* (Cantor 1886)

1. in God, *in natura naturans*, in the creating nature, where it is called *Absolute*,
2. *in concreto seu in natura naturata*, in perceptible things in created nature where he calls it *Transfinitum*,

3. *in abstracto*, that may be comprehended by human cognition in the form of *transfinite numbers*.

According to Cantor, philosophers earlier and contemporary can be divided depending on which of these forms they accept and which they do not. He himself fully and unconditionally agrees with all three.

The correspondence between Cantor and the leading Vatican theologian Cardinal Franzelin attests to the importance they both attached to the existence of the actual infinite (Cantor 1886). Cardinal Franzelin finally accepted Cantor's arguments and endorsed his theory: "When conceived in this way, so far as I can see at present, there is no danger to religious truths in your concept of *Transfinitum*." (Dauben 1990, 146).

Cantor was thus sure that he had built his mathematical theory on a solid metaphysical foundation, theologically grounded. He was convinced that his theory of infinite numbers was not just the only possible one, but also the only correct one. His unshakable confidence in his theory was infectious, until today most mathematicians do not doubt it.

4.4 Axiomatic set theories

Cantor's theory started to be more widely accepted after presenting a new proof of uncountability of real numbers, the diagonal argument, at the first meeting of Deutsche Mathematiker-Vereinung in 1891. At the beginning of the 20th century, set theory had been gradually accepted and enthusiastically developed by young and progressive mathematicians.

However, it was necessary to deal with the emerging paradoxes. The axiomatic systems determine exact rules for defining sets, establish deductive methods for proving and thus prevent paradoxes. Hilbert writes on this time in *On the Infinite*:

> We will carefully investigate fruitful definitions and deductive methods. We will nurse them, strengthen them, and make them useful. No one shall drive us out of the paradise which Cantor has created for us. (Hilbert 1926, p. 191).

The consensus was not complete. Some prominent mathematicians, such as Poincaré, Weyl or Wittgenstein, still opposed. However, the advantages brought by axiomatic set theory outweighed the objections of the time. Most of contemporary mathematics is performed in the *Zermelo-Fraenkel axiomatic system* usually with the *Axiom of Choice*, eventually, in some other system. They do no longer deal with the question of the existence of the actual infinite. The affirmative answer is expressed by the *Axiom of Infinity*: "The natural numbers form a set" or some its variant.

Another fundamental axiom, the *Power-Set Axiom*, guarantees that every set, finite or infinite set, is transparent, and that all its parts, individually and all together, are sets. These two axioms imply an infinite increasing sequences of ordinal and cardinal numbers. The *Continuum Hypothesis* would connect these two sequences. However, this has not happened and cannot happen. Indeed, the *Continuum Hypothesis*, like some other important statements, e.g., the *Axiom of Choice*, are independent theorems. They can neither be proved nor disproved from the axioms. Thus there is not a unique set theory, only its ground, which can be split into different branches, depending on one's choice. All that is certain is that one is working in a relatively consistent theory with respect to the the adopted axioms.

Moreover, according to Gődel's inconsistency theorems from 1931. any mathematical theory including axioms for arithmetics is consistent if and only if it is not complete, i.e. some valid statement is not provable.

Nobody can claim anymore, as Cantor did, the absolute truth of the set theory, and no one does. This question is left aside. It is as if inconsistency were more important than truth, and as if truth lost its meaning, certainly had less meaning than consistent description of the world. This is, after all, one of the sources of postmodern relativism.

Actual infinity belongs to mathematics, but God as its guarantor does not. If we reject it entirely, we considerably limit our research. But it is not necessary to accept in in Cantor's form. We can build set theory on another philosophical ground.

Chapter 5

The Alternative Set Theory

> *Wherever a vagueness occurs, look for a horizon and natural infinity that has caused this vagueness; then idealise this situation!* (Vopěnka 2022, 13).

Alternative Set Theory (AST) was created almost fifty years ago. Its first and best-known version was already published in 1979 (Vopěnka 1979). The question remains of whether this theory is still viable; which means whether it is worthwhile to deal with it from other than historical perspectives (Haniková 2022). From a purely mathematical point of view, AST is a special version of non-standard set theory. It was evaluated as "different from the usual set theories, much weaker, but mathematically rather interesting" (Holmes 2012, 624).

However, I believe that the main contribution of AST lies in the philosophical justification of its fundamental principles. (Trlifajová 2021). Vopěnka's intention was to restore the correspondence between mathematical notions and phenomena of the natural world, to bridge the gap between infinite mathematical objects and finite physical entities which appeared after the introduction of Cantor's set theory. This endeavour has also attracted the attention of scholars from other disciplines.

There are several reasons why Vopěnka's phenomenological ideas have not been sufficiently appreciated. The original book (Vopěnka 1979) is mostly mathematical except for the introduction. Explanations of new ideas are contained in other works, particularly in (Vopěnka 2001), that were written in Czech and have not been translated to English. Moreover, Vopěnka modified his theory several times, enriched, left blind alleys and looked for new ones. He searched for a better way to grasp fundamental phenomena he thought rational science had overlooked. While AST is partially axiomatised, the last version *New Infinitary Mathematics* (Vopěnka 2022) is open to interpretations both in a non-standard analysis and in a looser way in applied mathematics. The latter is more complicated and less comprehensible than AST.

Nevertheless, the themes that Vopěnka dealt with reappear with new intensity at the beginning of the new millennium, particularly in connection with vagueness theory, feasibility and alternatives to the classical set theory (Fletcher 2007; Bellotti 2008; Gaifman 2010; Dean 2018). Vopěnka was no longer in contact with the international scientific community, otherwise they could have inspired each other.

5.1 The horizon and the infinity

Where to find the infinite? Vopěnka provides a novel answer to the ancient question. He borrowed the term *horizon* from Husserl, not only in the sense of a visual horizon, but generally as "an indistinct boundary that separates the field of direct experience from that of indirect experience." (Vopěnka 1991, 117).

The horizon always depends on an observer and their point of view, however it appears outside in the real world. The horizon connects the first-person perspective, the subjective phenomenal account of an observation, and the third-person perspective, the objectified reflection of a situation behind the observation. Only through our ability to reflect do we know of its existence in the natural world.

In order to grasp the horizon mathematically, Vopěnka works in an axiomatic set theory AST. He uses common notions as set, empty set, subset, union, intersection, etc. He works with natural numbers $\mathbb{N}$ as the Von Neumann model of the Peano arithmetic and rational numbers $\mathbb{Q}$ as its quotient field. However, he adds the new notion of *semiset* and carefully distinguishes between sets and semisets. A *class* is any collection of things treated as an autonomous object. A *set* is a class that is clearly defined and its borders are precisely delimited. A *semiset A* is a vague subclass of a set bounded by a horizon. Many examples of semisets are given in the literature concerning vagueness and feasibility, Yessenin-Volpin's "feasible numbers", Wang's "small numbers", "orange objects" (Dummett 1975), "inhabitants of a small town" (Dean 2018), "walking distance" (Gaifman 2010). Their existence is in AST guaranteed by the axiom.

Axiom of Proper Semisets. There is a proper semiset.

The infinite is comprehended as a phenomenon involved in the observation of large, incomprehensible classes. If a part of a class A is not a set, it is a semiset, then the class A is called *infinite*. Consequently, all parts of a *finite* class A are sets.

$$Fin(A) \Leftrightarrow (\forall B)(B \subseteq A \Rightarrow Set(B))$$

In this figurative sense, there are infinitely many people in the world, grains of sand in a heap, flowers in a blooming meadow, ancestors of a particular person, etc. It follows from the *Axiom of Proper Semisets* that there are infinite sets.

5.2 The horizon and the countability

One type of infinity is countable infinity. It is a representation of a path toward the horizon such as the sleepers on the railway tracks leading to the horizon, the ever-diminishing reflections in two mirrors facing each other or one's ancestors. A *countable* class $(C, \leq)$ is defined as an infinite, linearly ordered class such that all its initial segments are finite

$$(\forall x)(x \in C \Rightarrow Fin(\{y \in C; y \leq x\})).$$

Finite natural numbers $\mathbb{N}$ lie in front of the horizon, they are small, accessible, feasible. The class of *finite natural numbers* $\mathbb{FN}$ is countable.

$$\mathbb{FN} = \{n \in \mathbb{N}; Fin(n)\} \subseteq \mathbb{N}.$$

A path toward a horizon does not stop exactly on the horizon but continuous at least a little further. A countable class can be prolonged to an infinite set whose elements have the same set-theoretic properties. Therefore AST accepts the axiom.

> *Prolongation Axiom:* A countable function F can be extended to a set function F such that $F \subseteq f$.

The countable class of finite natural numbers $\mathbb{FN}$ can be prolonged to a set whose elements have all properties of natural number. This set is also a natural number α such that $\mathbb{FN} \subseteq \alpha$, hence α is and infinite natural number. Metaphorically, α is beyond the horizon of our observation.

5.3 The horizon and the continuum

A continuum is understood as a phenomenon that we perceive when observing a large but remote class, and we are unable to distinguish its individual elements because they lie beyond the horizon of our observational capabilities. This is due to the imperfection of our senses. However, this weakness is our advantage. For this reason we perceive the given structure as continuous. For example, a heap composed of grains of sand. Its surface appears continuous from a sufficiently great distance because we are unable to distinguish the individual grains, they are indiscrnible.

This concept of continuum again joins the the objective existence of an underlying structure and the subjective point of view of an observer, who determines the indiscernibility relation.

Mathematically, a continuum is described as an infinite set that forms that underlying structure with an indiscernibility relation defined on it. Two elements of this structure are indiscernible if their difference is beyond the horizon of our observational capacity. The small semisets of mutually indiscernible elements lying on the horizon are called *monads*. The continuum is the infinite union of all monads.

The straight line is an example of continuum. When we choose a coordinate 0 and 1 then the line can be marked by the rational numbers $\mathbb{Q}$ as coordinates. This is the underlying structure. We define for $x, y \in \mathbb{Q}$:

- x is *infinitely small* if $(\forall n \in \mathbb{FN})(|x| < \frac{1}{n})$, "$x$ is beyond the horizon of depth";
- x is *infinitely great* if $(\forall n \in \mathbb{FN})(|x| > n)$, "$x$ is beyond the hor. of distance";
- x is *bounded* if $(\exists n \in \mathbb{FN})(|x| < n)$, "$x$ is in front of the horizon of a distance";
- $x, y \in \mathbb{BQ}$ are *indiscernible*, $x \doteq y \Leftrightarrow (\forall n)(n \in \mathbb{FN} \Rightarrow |x - y| < \frac{1}{n})$.
- A monad $Mon(x) = \{y \in Q; y \doteq x\}$ represents a *point* of a line.

The definition of infinitely small numbers assumes that the two horizons of a "depth" and of a "distance" correspond with each other. This agrees with Pascal's concept of two infinities: the infinitely great and the infinitely small correspond to one another. (Pascal 1866, 288).

5.4 Where is the horizon?

Vopěnka had always avoided directly answering this key question. While in (Vopěnka 1979) he works with an ideal mathematical horizon, later he wished to let this question open and to create a theory which is valid both for real and ideal entities.

If we accept a concrete number as bounding the horizon, then AST encounters the same problems as the feasibility theory that is only "almost consistent". If we assume that the horizon is mathematically unbounded then the finite natural numbers $\mathbb{FN}$ form the same structure as standard natural numbers in a non-standard model. (Bellotti 2008).

In the real world, the horizon is always bounded in some way by a concrete number. There are many different horizons depending on the observer, the object and the unit of observation. Similarly, the indiscernibility forming a continuum is generally not transitive, thus monads are not always disjunctive, they partially overlap.

The idealized horizon is unbounded, finite natural numbers $\mathbb{FN}$ form a model of the Peano Arithmetic. Consequently, the indiscernibility is transitive, it is an equivalence relation. Hence monads are disjunctive.

When we define the class of *bounded numbers* $\mathbb{BQ} = \{x \in \mathbb{Q}; (\exists n \in FN)|x| < n\}$ then by its factorization modulo the indiscernibilty $\doteq$ we get the structure of real numbers, an Archimedean, linearly ordered, dense, complete field.

$$\mathbb{BQ}/\doteq \quad = \quad \mathbb{R}.$$

Real numbers are represented as disjoint monads of indiscernible rational numbers.

5.5 The abstraction and the idealization

To get from real-world phenomena to their mathematical representations requires always abstraction and often idealization. Abstraction can be described as a representation that highlights some properties and disregards others, while idealization allows to change some aspects of object to obtain its ideal limit form, it is a "deliberate misrepresentation of some aspects of the world" (Levy 2021).

Two directions are present in the process of idealization: one ascending from the real world to the ideal, the other descending back. That is the reason why results of infinite mathematics are applicable in the real world.

> The idealities which are assumed by the second process of idealization serve as a priori "guides" in the further determination of those vague and inexact empirical entities within the manifold of intuition. They are geometrically determinable to precisely the degree that they "participate" in the pure geometrical forms which they motivate. (Garrison 1986, 331)

Vopěnka describes carefully the process of *abstraction* although he does not called it that (Vopěnka 2001). All his notions such as set, semiset, finite set, infinite class, countable class, indiscernibility, arose as *abstractions* of phenomena of the real world. A countable class is the abstraction of a path toward a horizon. Mathematical statements can be expressed already using these notions and this novel perspective inspires further research.

The second step is the *idealization*. Finite natural numbers $\mathbb{FN}$ represent the idealization of a countable class. They are not dependent on a particular situation nor bounded by a concretee number. This is a deliberate misrepresentation, the limitation of which we must be aware. The reward is that we work in a consistent mathematical system from which we can always return to physical objects.

> We apply the mathematical results in the real world in the same way in which the results of ancient geometry have been applied ever since its origins. That is by substituting the geometric horizon with the horizon limiting the human looks into the natural real world while evidencing inevitable distortions this brings. (Vopěnka 2022, 154).

Chapter 6

Appendix

6.1 Bolzano's measurable numbers: are they real?

Russ, S., Trlifajová, K. (2016) In: M. Zack & E. Landry, *Research in History and Philosophy of Mathematics*. Basel: Birkhäuser, 39–56.
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6.2 Bolzano's Infinite Quantities.

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6.3 Sizes of Countable Sets.

Trlifajová, K. (2024) *Philosophia Mathematica*. 32(1), 82–114.
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6.4 Theological reasoning of Cantor's set theory

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6.5 Infinity and Continuum in the Alternative Set Theory

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