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HABILITATION THESIS

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Differential graded rings and their applications

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To Jenya with love, and in memory of our grandparents.

Contents

Chapter 0

Introduction

This habilitation thesis consists of this introduction, as well as the following eight original research papers:

(P1) Liran Shaul,
Hochschild cohomology commutes with adic completion,
Algebra & Number Theory 10-5, 1001-1029 (2016),
doi:10.2140/ant.2016.10.1001.

(P2) Liran Shaul,
The Cohen-Macaulay property in derived commutative algebra,
Transactions of the of the American Mathematical Society, 373 (2020), no. 9, 6095–6138,
doi:10.1090/tran/8099.

(P3) Liran Shaul,
Koszul complexes over Cohen-Macaulay rings,
Advances in Mathematics, Volume 386, 6 August 2021,
doi:10.1016/j.aim.2021.107806.

(P4) Liran Shaul,
Sequence-regular commutative DG-rings,
Journal of Algebra, Volume 647, 1 June 2024, Pages 400-435,
doi:10.1016/j.jalgebra.2024.02.034.

(P5) Liran Shaul, Jordan Williamson,
Lifting (co)stratifications between tensor triangulated categories,
Israel Journal of Mathematics, in press, 2024,
doi:10.1007/s11856-023-2578-5.

(P6) Isaac Bird, Liran Shaul, Prashanth Sridhar, Jordan Williamson,
Finitistic dimensions over commutative DG-rings,
preprint at arxiv:2204.06865.

(P7) Liran Shaul,
The finitistic dimension conjecture via DG-rings,
preprint at arxiv:2209.02068.

(P8) Liran Shaul, with an appendix by Tsutomu Nakamura and Peder Thompson,
Acyclic complexes of injectives and finitistic dimensions,
preprint at arxiv:2303.08756.

The common theme of all these papers is the notion of a differential graded ring (which will always be abbreviated as DG-ring). A DG-ring is an associative unital ring which is also a cochain complex, such that the multiplication of the ring and the differential of the complex are compatible with each other (see precise definition in Section 0.1 below). In particular, any ring is a DG-ring, making the category of DG-rings a natural extension of the category of rings. The aim of this thesis, carried out in these papers, is to make a detailed study of DG-rings and their derived categories, and to apply these results to prove new facts about rings.

More specifically, we will always focus on non-positively graded (in cohomological grading) DG-rings, also known in the literature as connective DG-algebras. From a homotopy-theoretic point of view, such DG-rings carry naturally a model structure, and this model category is Quillen-equivalent to the model category of simplicial rings (see [31, Theorem 1.1(3)]). This implies that one can view these DG-rings as a model for derived rings, and allows replacing a map of rings by a semi-free DG-algebra resolution, which behaves better from a homological point of view. This is analogous to the situation in ordinary homological algebra, where the category of modules is embedded into the category of cochain complexes, which allows one to replace modules by projective resolutions, and define derived functors. We shall now provide some preliminary background that will put the results of this thesis into perspective.

0.1 DG-rings

A DG-ring A is an associative unital graded ring

$$A = \bigoplus_{n=-\infty}^{\infty} A^n.$$

It is further equipped with a degree $+1$ map, the differential of A, which is a $\mathbb{Z}$-linear map $d : A \to A$, such that $d \circ d = 0$, and such that the Leibnitz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} \cdot a \cdot d(b)$$

holds for any pair of homogeneous elements $a \in A^m$, $b \in A^n$, with $m, n \in \mathbb{Z}$, where $\deg(a) = m$. The earliest mention of DG-rings we were able to find in the literature is in the paper [11] by Cartan from 1954. We say that A is non-positive if $A^n = 0$ for all $n > 0$. We will always assume that DG-rings are non-positive, so everywhere below when we write DG-ring we mean a non-positive DG-ring. Any ring A can be considered as a DG-ring by letting $A^n = 0$ for all $n \neq 0$, and taking $d = 0$.

The zeroth cohomology $\mathrm{H}^0(A)$ of a DG-ring A is a ring, and there is a map of DG-rings $A \to \mathrm{H}^0(A)$. The DG-ring A and the ring $\mathrm{H}^0(A)$ are closely related, and many of the results presented below build upon the connection between these two.

To gain a better understanding about the relations between A and $\mathrm{H}^0(A)$, we must discuss DG-modules and derived categories. Triangulated and derived categories, which are the heart of modern homological algebra were first introduced by Verdier [36], following ideas of Grothendieck [13], and independently by Puppe [28].

Left DG-modules over a DG-ring A, which generalize left modules and complexes of left A-modules, are by definition left graded A-modules $M = \bigoplus_{n=-\infty}^{\infty} M^n$ with a differential $d : M \to M$ of degree $+1$ which satisfies a Leibnitz rule. In particular, for $n \in \mathbb{Z}$, it follows that the cohomology $\mathrm{H}^n(M)$ is an $\mathrm{H}^0(A)$-module. If $f : M \to N$ is a map of left DG-modules over A, it is called a quasi-isomorphism if $\mathrm{H}^n(f) : \mathrm{H}^n(M) \to \mathrm{H}^n(N)$ is an isomorphism for all $n \in \mathbb{Z}$.

Considering the category $\mathrm{DGMod}(A)$ of all DG-modules over A, and formally inverting all quasi-isomorphisms one obtains the derived category of left DG-modules over A, denoted by $\mathsf{D}(A)$. It is a triangulated category. We refer the reader to the recent book [40] for a modern treatment of derived categories of DG-rings.

One key reason for the intimate relation between A and $\mathrm{H}^0(A)$ arises from the theory of t-structures. Recall that a t-structure on a triangulated category $\mathcal{T}$, first defined by Beilinson, Bernstein and Deligne in [8], is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of subcategories of $\mathcal{T}$, such that for any $X \in \mathcal{T}^{\leq 0}$ and any $Y \in \mathcal{T}^{\geq 0}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[-1]) = 0$ (where $[-1]$ is the inverse of the shift functor of $\mathcal{T}$), such that $\mathcal{T}^{\leq 0}$ is closed under shift and $\mathcal{T}^{\geq 0}$ is closed under the inverse of shift, and such that for any $Z \in \mathcal{T}$, there is a distinguished triangle

$$X \to Z \to Y[-1] \to X[1]$$

with $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 0}$.

It is shown in [8, Theorem 1.3.6] that the intersection $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category, called the heart of the t-structure. If A is a ring, the derived category of left A-modules $\mathsf{D}(A)$ carries naturally a canonical t-structure given by

$$\mathsf{D}(A)^{\leq 0} := \{M \in \mathsf{D}(A) \mid \mathrm{H}^n(M) = 0, \forall n > 0\},$$

and

$$\mathsf{D}(A)^{\geq 0} := \{M \in \mathsf{D}(A) \mid \mathrm{H}^n(M) = 0, \forall n < 0\}.$$

The heart of this t-structure is simply $\mathrm{Mod}(A)$, the category of left A-modules.

More generally, if A is a non-positive DG-ring (we emphasize non-positivity here, because this fact is not true otherwise), then, as follows from the discussion in [40, Section 7.3], it is still the case that

$$\mathsf{D}(A)^{\leq 0} := \{M \in \mathsf{D}(A) \mid \mathrm{H}^n(M) = 0, \forall n > 0\},$$

and

$$\mathsf{D}(A)^{\geq 0} := \{M \in \mathsf{D}(A) \mid \mathrm{H}^n(M) = 0, \forall n < 0\}$$

define a t-structure on the derived category $\mathsf{D}(A)$ of left DG-modules over A. The important fact that connects A and $\mathrm{H}^0(A)$ for A non-positive is that the heart of this t-structure on $\mathsf{D}(A)$ is equal to the abelian category $\mathrm{Mod}(\mathrm{H}^0(A))$. This is used implicitly throughout this thesis to prove many new results about DG-rings. Our recent paper [35], which is not a part of this thesis, offers a systematic study of other relations between $\mathsf{D}(A)$ and $\mathsf{D}(\mathrm{H}^0(A))$. The paper [P5] discussed in detail Section 0.4 studies an even more intimate connection between $\mathsf{D}(A)$ and $\mathsf{D}(\mathrm{H}^0(A))$ in some specific cases.

To study DG-rings further, it is essential to introduce some finiteness condition. We say that a DG-ring A is left noetherian if the ring $\mathrm{H}^0(A)$ is left noetherian, and for any $n < 0$, the left $\mathrm{H}^0(A)$-module $\mathrm{H}^n(A)$ is finitely generated. One can justify this definition using the derived Bass-Papp theorem we proved in [33, Theorem 6.6]. Right noetherian DG-rings are defined similarly. We say that A has bounded cohomology if $\mathrm{H}^n(A) = 0$ for all $n \ll 0$. Many results of this thesis are proved in the case where A is left noetherian (or two-sided noetherian) and has bounded cohomology.

We further often restrict attention to triangulated subcategories of $\mathsf{D}(A)$. The full triangulated subcategory $\mathsf{D}^-(A)$ is the derived category of bounded above DG-modules, composed of all $M \in \mathsf{D}(A)$ such that $\mathrm{H}^n(M) = 0$ for $n \gg 0$. Similarly, $\mathsf{D}^+(A)$, the derived category of bounded below DG-modules is the full triangulated subcategory of $M \in \mathsf{D}(A)$ such that $\mathrm{H}^n(M) = 0$ for $n \ll 0$. Finally, the derived category of bounded DG-modules is $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{D}^-(A) \cap \mathsf{D}^+(A)$. In case A is left noetherian, we further consider the derived category $\mathsf{D}_{\mathrm{f}}(A)$ of DG-modules with finitely generated cohomology, that is, $M \in \mathsf{D}_{\mathrm{f}}(A)$ if for all $n \in \mathbb{Z}$ it holds that $\mathrm{H}^n(M)$ is a finitely generated left $\mathrm{H}^0(A)$-module. One further consider combinations of these finiteness conditions, setting for instance

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A) = \mathsf{D}_{\mathrm{f}}(A) \cap \mathsf{D}^{\mathrm{b}}(A).$$

The majority of this thesis, namely, the papers [P1,P2,P3,P4,P5,P6] studies commutative DG-rings. These are, by definition, non-positive DG-rings that satisfy the additional equation $b \cdot a = (-1)^{\deg(a)\cdot\deg(b)} \cdot a \cdot b$, for any pair of homogeneous elements $a, b \in A$, and moreover $a^2 = 0$ if $a \in A^n$ with n being odd. Note that the latter assumption follows from the former if $2 = 1 + 1$ is invertible in A. The commutative assumption implies that we may identify left and right DG-modules, obtaining a single derived category of DG-modules over $\mathsf{D}(A)$. Moreover, this triangulated category has an extra structure, because if $M, N \in \mathsf{D}(A)$ then their derived tensor product $M \otimes^{\mathrm{L}}_A N$ also belongs to $\mathsf{D}(A)$. This gives $\mathsf{D}(A)$ the structure of a tensor triangulated category, in the sense of [3].

Many methods of commutative algebra can be applied in this setting. For instance, following [38, Section 4], one can localize a commutative DG-ring at a prime in $\mathrm{H}^0(A)$: If $\bar{\mathfrak{p}} \in \operatorname{Spec}(\mathrm{H}^0(A))$, letting $\mathfrak{p}$ be its preimage in A^0, one sets $A_{\bar{\mathfrak{p}}} = A^0_{\mathfrak{p}} \otimes_{A^0} A$. One can also form the (derived) $\bar{\mathfrak{a}}$-adic completion of A with respect to an ideal $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$. We developed the theory of derived adic completion of commutative DG-rings in the paper [34] which is not part of this thesis.

If A is a commutative noetherian (which is the same as left noetherian) DG-ring such that the commutative noetherian ring $\mathrm{H}^0(A)$ is a local ring with maximal ideal $\bar{\mathfrak{m}}$, we say that $(A, \bar{\mathfrak{m}})$ is a noetherian local DG-ring. It follows that if A is a commutative noetherian DG-ring and $\bar{\mathfrak{p}} \in \operatorname{Spec}(\mathrm{H}^0(A))$, then $(A_{\bar{\mathfrak{p}}}, \bar{\mathfrak{p}} \cdot \mathrm{H}^0(A_{\bar{\mathfrak{p}}}))$ is a noetherian local DG-ring.

0.2 Cohen-Macaulay and Regular rings

If V is a finite dimensional vector space over a field $\mathbb{K}$, there are two approaches to define the dimension of V, a geometric approach, and an algebraic approach. The geometric approach considers chains of vector subspaces of V. For instance, if $\mathbb{K} = \mathbb{R}$, and $V = \mathbb{R}^3$, any proper vector subspace of V is either the point $\{0\}$, a line L passing though the origin, or a plane P passing through the origin. A maximal chain of proper inclusions of such subspaces is thus of the form $0 \subsetneq L \subsetneq P \subsetneq V$. As this chain has length 3, this geometric approach to dimension suggests that $\dim(V) = 3$.

Alternatively, an algebraic approach to dimension is also possible. In this approach, we start with the vector space V, and keep imposing new linearly independent linear equations on it, until the only point which satisfies these linear equations is the origin. The number of such independent linear equations we may impose provides an algebraic approach to dimension. Basic results in linear algebra show that these two approaches always yield the same number, the dimension of V.

Importing these ideas to commutative algebra and algebraic geometry leads us to the notion of a Cohen-Macaulay ring. We work locally, and consider a commutative noetherian local ring $(A, \mathfrak{m})$, which geometrically corresponds to the behavior of a given algebraic variety V at a point $p \in V$. The geometric approach to dimension in this context is using the notion of Krull dimension, first defined in [20] by Krull in 1937. This is defined to be the length n of a maximal chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}$ in A. It is denoted by $\dim(A)$. The algebraic approach to dimension in this context is using the notion of depth. An element $x \in \mathfrak{m}$ is called an A-regular element if it is not a zero-divisor on A. More generally, a sequence of elements $x_1, \dots, x_n \in \mathfrak{m}$ is called an A-regular sequence if x_1 is A-regular, and the sequence $x_2, \dots, x_n$ is A/x_1-regular. The maximal length n of an A-regular sequence is called the depth of A, and is denoted by $\operatorname{depth}(A)$. This was first defined by Auslander and Buchsbaum [2] in 1957, where the authors also proved [2, Proposition 1.3] that the inequality $\operatorname{depth}(A) \leq \dim(A)$ always holds. When this inequality is an equality, so that

$$\operatorname{depth}(A) = \dim(A), \tag{CM1}$$

we are in a situation where the geometric and algebraic approaches to dimension coincide, and then A is called a **Cohen-Macaulay ring**. Generalizing this notion from rings to DG-rings is one of the main topics of this thesis. We shall give 6 different statements which are equivalent to being a Cohen-Macaulay ring (some of them only works in specific situations), and generalize all of them to the DG setting, the first of them being (CM1) above.

Give a commutative ring A, and an ideal $I \subseteq A$, the I-torsion functor $\Gamma_I : \operatorname{Mod}(A) \to \operatorname{Mod}(A)$ is give by $\Gamma_I(-) := \varinjlim \operatorname{Hom}_A(A/I^n, -)$. It is a left exact additive functor, and its right exact functor $\mathrm{R}\Gamma_I$ is called the derived I-torsion functor. Its cohomologies are the classical derived functor $\mathrm{H}^n_I(-)$, commonly known as local cohomology or cohomology supported at $V(I)$. It was first defined and studied by Grothendieck, see [14].

Let $(A, \mathfrak{m})$ be a commutative noetherian local ring. According to [14, Corollary 3.10], there is an equality $\operatorname{depth}(A) = \min\{n \in \mathbb{Z} \mid \mathrm{H}^n_{\mathfrak{m}}(A) \neq 0\}$. On the other hand, by [14, Proposition 6.4], it holds that $\dim(A) = \max\{n \in \mathbb{Z} \mid \mathrm{H}^n_{\mathfrak{m}}(A) \neq 0\}$. It follows that A is Cohen-Macaulay if and only if its local cohomology is concentrated in a single degree, that is, when

$$\operatorname{amp}(\mathrm{R}\Gamma_{\mathfrak{m}}(A)) := \sup\{n \mid \mathrm{H}^n(\mathrm{R}\Gamma_{\mathfrak{m}}(A)) \neq 0\} - \inf\{n \mid \mathrm{H}^n(\mathrm{R}\Gamma_{\mathfrak{m}}(A)) \neq 0\} = 0 \tag{CM2}$$

A noetherian local ring $(A, \mathfrak{m})$ is called Gorenstein if the A-module A has finite injective dimension over itself. By [5, Corollary 3.4], any Gorenstein ring is Cohen-Macaulay. Bass further generalized this observation in [5], and noted noted that if $(A, \mathfrak{m})$ is Cohen-Macaulay, then there exist a non-zero finitely generated A-module of finite injective dimension. He then stated that the converse is convincible, and the question whether a local ring ring which admits a non-zero finitely generated module of finite injective dimension is Cohen-Macaulay became known as the Bass conjecture. This was settled affirmatively by Peskine and Szpiro in [25, Corollary 4.13], so that $(A, \mathfrak{m})$ is Cohen-Macaulay if and only if

$$\exists M \in \operatorname{Mod}(A), M \ncong 0, M \text{ is finitely generated}, \operatorname{inj}\dim_A(M) < \infty. \tag{CM3}$$

An important property of Cohen-Macaulay rings is that they often enjoy a good duality theory. Following Grothendieck's work recorded by Hartshorne in [13], recall that a dualizing complex R over a commutative noetherian ring A is a bounded complex R with finitely generated cohomology, such that $\operatorname{inj}\dim_A(R) < \infty$, and such that the natural map $A \to \mathrm{R}\operatorname{Hom}_A(R, R)$ is an isomorphism in the derived category $\mathsf{D}(A)$. Dualizing complexes give rise to duality: if R is a dualizing complex over A, for any finitely generated A-module M, the natural map

$$M \to \mathrm{R}\operatorname{Hom}_A(\mathrm{R}\operatorname{Hom}_A(M, R), R)$$

is an isomorphism in the derived category $\mathsf{D}(A)$.

It follows from Grothendieck's local duality theorem [13, Theorem V.6.2], that if $(A, \mathfrak{m})$ is a noetherian local ring with a dualizing complex R, then

$$\operatorname{amp}(R) := \sup\{n \mid \mathrm{H}^n(R) \neq 0\} - \inf\{n \mid \mathrm{H}^n(R) = 0\} = \dim(A) - \operatorname{depth}(A).$$

We remark that we generalized the local duality theorem to the DG setting in [33, Theorem 7.26], but the paper [33] is not a part of this thesis.

Not every Cohen-Macaulay local ring admits a dualizing complex. A counterexample is given in [24, Section 6, Example 1]. However, most noetherian local rings that arise naturally in commutative algebra and algebraic geometry admit dualizing complexes. For a noetherian local ring $(A, \mathfrak{m})$ with a dualizing complex R, it follows from local duality that A is Cohen-Macaulay if and only if

$$\operatorname{amp}(R) := \sup\{n \mid \mathrm{H}^n(R) \neq 0\} - \inf\{n \mid \mathrm{H}^n(R) = 0\} = 0, \tag{CM4}$$

in other words, if and only if the dualizing complex R is a finitely generated A-module (up to isomorphism and shifting, two operations which preserve the property of being a dualizing complex).

The class of Cohen-Macaulay rings contains the particularly important class of regular rings. If $(A,\mathfrak{m})$ is a noetherian local ring, recall that $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated vector space over the residue field $\Bbbk = A/\mathfrak{m}$, called the cotangent space of A. The dimension of the cotangent space satisfies the inequality

$$\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(A).$$

In case there is an equality $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$, the ring A is called a regular local ring. As shown in [2, 32], there are two other properties of a noetherian local ring which are equivalent to it being regular. The first is that the maximal ideal $\mathfrak{m}$ is generated by an A-regular sequence. This also implies that it is Cohen-Macaulay. The second is that A has finite global dimension. From the latter it follows that the localization of a regular local ring at a prime ideal is again regular. In [P4] we generalize the property of the maximal ideal being generated by a regular sequence to the DG setting, and moreover show that this property is also stable under localization.

Suppose now that $(A,\mathfrak{m})$ is a regular local ring, that $(B,\mathfrak{n})$ is a noetherian local ring, and that there is a finite injective map of local rings $A \to B$. In his unpublished master thesis, Hironaka showed that in this situation B is Cohen-Macaulay if and only if

$$B \text{ is a free } A\text{-module.} \tag{CM5}$$

Next we discuss an important DG-ring that arises in commutative algebra, the Koszul complex. For an element $a \in \mathfrak{m}$ in a noetherian local ring $(A,\mathfrak{m})$, it is given by the complex

$$K(A;a) := 0 \to A \xrightarrow{\cdot a} A \to 0.$$

concentrated in cohomological degrees $-1, 0$. It is interesting to note that that the cohomologies of it satisfy that $\mathrm{H}^0(K(A;a)) = A/a$ and

$$\mathrm{H}^{-1}(K(A;a)) = \{x \in A \mid a \cdot x = 0\}.$$

It follows that $K(A;a) \cong A/a$ if and only if a is an A-regular element. More generally, one may define the Koszul complex $K(A;a_1,\dots,a_n)$ associated to any finite sequence of elements $a_1,\dots,a_n \in \mathfrak{m}$ in a noetherian local ring $(A,\mathfrak{m})$, and one can show (see for instance [21, Theorem 16.5]) that $K(A;a_1,\dots,a_n) \cong A/(a_1,\dots,a_n)$ if and only if $a_1,\dots,a_n$ is an A-regular sequence. As we explain in [P3], one may consider $K(A;a_1,\dots,a_n)$ as a non-abelian derived functor of the operation $A \mapsto A/(a_1,\dots,a_n)$. In particular, the ordinary quotient $A/(a_1,\dots,a_n)$ is particularly well behaved in the case where $a_1,\dots,a_n$ is a regular sequence. For example, as shown in [21, Theorem 17.3], if $(A,\mathfrak{m})$ is a Cohen-Macaulay local ring and $a_1,\dots,a_n \in \mathfrak{m}$ is an A-regular sequence, then $A/(a_1,\dots,a_n)$ is also a Cohen-Macaulay ring. The paper [P3] contains a far reaching generalization of this fact, which is valid for any finite sequence of elements in a Cohen-Macaulay ring.

We discussed above five different characterizations of Cohen-Macaulay rings, [CM1-CM5]. These will be generalized to the DG setting in [P2,P3]. More specifically, given a commutative noetherian local DG-ring $(A, \bar{\mathfrak{m}})$ with bounded cohomology, we define the notion of a local-Cohen-Macaulay DG-ring in [P2, Definition 4.2] generalizing (CM2). In [P2, Corollary 5.5] we show that this is equivalent to a DG version of (CM1). A version of the Bass conjecture, (CM3) in this setting is established in [P2, Theorem 5.22]. Using the notion of a dualizing DG-module, we obtain a DG-version of (CM4) in [P2, Theorem 4.1]. Finally, a DG version of Hironaka's miracle flatness theorem (CM5), is obtained in [P3, Corollary 5.7]. A sixth characterization of Cohen-Macaulay rings and its generalization to the DG setting will be discussed in Section 0.5.

0.3 Completion and Hochschild cohomology

Hochschild cohomology is the most prominent cohomology theory for associative algebras, first introduced by Gerhard Hochschild in his paper [15] in 1945. In commutative algebra, it was shown in [16] that for smooth algebras, one can identify their Hochschild cohomology with the wedge products of their derivations.

When constructing the Hochschild cohomology of a flat commutative $\mathbb{K}$-algebra A over a commutative base ring $\mathbb{K}$, as part of the construction, one must consider the enveloping algebra $A \otimes_{\mathbb{K}} A$. The Hochschild cohomology modules of A are then given by

$$\mathrm{HH}^n(A/\mathbb{K}) := \mathrm{Ext}^n_{A\otimes_{\mathbb{K}} A}(A, A).$$

For most flat commutative algebras that one encounters in commutative algebra and its applications, this is not a problem, because in most situations of interest, the map $\mathbb{K} \to A$ is of finite type, or at least essentially of finite type (that is, a localization of a map of finite type). In that case, if $\mathbb{K}, A$ are noetherian and $\mathbb{K} \to A$ is essentially of finite type, the enveloping algebra $A \otimes_{\mathbb{K}} A$ is also noetherian.

There is however one exception to this, namely, algebras that arise from the adic completion operation. The basic example is where $\mathbb{K}$ is a field, and $A = \mathbb{K}[[x]]$, the power series ring over a field, which is the (x)-adic completion of the polynomial ring $\mathbb{K}[x]$. Such rings also arise naturally in algebraic geometry when one considers formal schemes instead of ordinary schemes. The problem with developing a theory of Hochschild cohomology of such algebras is that in this situation the enveloping algebra $A \otimes_{\mathbb{K}} A$ is usually non-noetherian. This is the case for example when $A = \mathbb{K}[[x]]$ where $\mathbb{K}$ is a field of characteristic 0.

This basic problem is related to the fact that in the category of formal schemes, the fiber product of is not locally represented by the tensor product, but instead, by the complete tensor product. If $\mathbb{K}$ is a base commutative noetherian ring, we consider $\mathbb{K}$-algebras A with the property that there exist an ideal $\mathfrak{a} \subseteq A$ such that A is $\mathfrak{a}$-adically complete, and such that the quotient $A/\mathfrak{a}$ is a finite type $\mathbb{K}$-algebra. In that situation, one says that $(A, \mathfrak{a})$ is a formally of finite type adic $\mathbb{K}$-algebra. If $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$ are two formally of finite type adic $\mathbb{K}$-algebras, their completed tensor product is the formally of finite type $\mathbb{K}$-algebra

$$A \widehat{\otimes}_{\mathbb{K}} B = \Lambda_{\mathfrak{a}\otimes_{\mathbb{K}} B + A\otimes_{\mathbb{K}} \mathfrak{b}}(A \otimes_{\mathbb{K}} B).$$

Suppose now that $\mathbb{K}$ is noetherian and that $(A, \mathfrak{a})$ is a formally of finite type flat adic $\mathbb{K}$-algebra (for example, $A = \mathbb{K}[[x]]$, and $\mathfrak{a} = (x)$). Then one defines the complete enveloping algebra of $(A, \mathfrak{a})$ over $\mathbb{K}$ by $A \widehat{\otimes}_{\mathbb{K}} A$, and it is possible to show that it is noetherian. These considerations led Buchweitz and Flenner in [9] to define the complete Hochschild cohomology of A over $\mathbb{K}$ by replacing the ordinary enveloping algebra with the complete enveloping algebra. Thus, assuming that $(A, \mathfrak{a})$ is a flat formally of finite type adic $\mathbb{K}$-algebra, one may consider

$$\widehat{\mathrm{HH}}^n(A/\mathbb{K}) := \mathrm{Ext}^n_{A\widehat{\otimes}_{\mathbb{K}} A}(A, A).$$

Working locally, suppose now that $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are complete noetherian local rings, and that there is a local homomorphism $A \to B$, with the property that the induced map $A/\mathfrak{m} \to B/\mathfrak{n}$ is an isomorphism. In this situation, Buchweitz and Flenner observed in [9, Section 3] that there is a natural map

$$\widehat{\mathrm{HH}}^n(B/A) \to \mathrm{HH}^n(B/A)$$

between the complete Hochschild cohomology of B over A to the ordinary Hochschild cohomology of B over A. They then asked if this map is an isomorphism.

To understand the difficulty in answering this question, observe that in the construction of $\widehat{\mathrm{HH}}^n(B/A)$, when considering the complete enveloping algebra

$$\Lambda_{\mathfrak{n}\otimes_A B + B\otimes_A \mathfrak{n}}(B \otimes_A B),$$

while in the situation above this algebra is noetherian, it is given as the adic completion of the ring $B \otimes_A B$ which is usually non-noetherian. But the classical theory of adic completion of commutative algebras only works for noetherian rings. For instance, while if A is noetherian the completion map $A \to \widehat{A}$ is always flat (see for instance [21, Theorem 8.8]), when considering the completion of non-noetherian rings this is usually not the case. For example, if $\mathbb{K}$ is a field of characteristic 0 and $A = \mathbb{K}[[x]]$, it is shown in [39, Theorem 7.2] that the completion map $A \otimes_{\mathbb{K}} A \to A \widehat{\otimes}_{\mathbb{K}} A$ is not flat.

To tackle this problem, the notion of a weakly proregular ideal was developed. It was first implicitly used in [14], and first explicitly mentioned in [1, Correction]. We further developed it in [26], where it was shown that this is the precise condition needed for local cohomology and derived completion to behave well as in the noetherian case. In a noetherian ring, any ideal is weakly proregular, but there are interesting examples also in non-noetherian situations.

In the paper [P1] we used this notion to solve the question raised by Buchweitz and Flenner in [9]. First, we showed in [P1, Theorem 2.6] that if $\mathbb{K}$ is noetherian, $(A, \mathfrak{a})$ is a formally of finite type flat adic $\mathbb{K}$-algebra, then the ideal

$$I := \mathfrak{a} \otimes_{\mathbb{K}} A + A \otimes_{\mathbb{K}} \mathfrak{a} \subseteq A \otimes_{\mathbb{K}} A$$

is weakly proregular. We then used DG-ring techniques, by taking a semi-free DG-algebra resolution of the completion map $A \otimes_{\mathbb{K}} A \to A \widehat{\otimes}_{\mathbb{K}} A$ to gain better understanding of such completions, and used this in [P1, Corollary 4.5] to obtain a positive resolution to the question of Buchweitz and Flenner.

0.4 Localizing and colocalizing subcategories

A guiding problem in the theory of triangulated categories in general, and tensor-triangulated categories in particular, is to classify all objects in a given triangulated category. Usually, this problem is too difficult, so instead one studies approximations of this problem, where one attempts to classify objects up to extensions, retracts and products or coproducts. This leads to the problem of classifying all thick, localizing and colocalizing subcategories of a given triangulated category $\mathcal{T}$. Assume that $\mathcal{T}$ is a triangulated category which admits all products and coproducts. The derived category $\mathsf{D}(A)$ where A is a ring or a DG-ring is an example of such a $\mathcal{T}$. Recall that a full triangulated subcategory of $\mathcal{S}$ of $\mathcal{T}$ is called a localizing subcategory if it closed under coproducts. It is called a colocalizing subcategory if it is closed under products. It is called a thick subcategory if it is closed under direct summands.

Results classifying localizing, colocalizing and thick subcategories first originated in homotopy theory, where Hopkins and Smith classified in [18] thick subcategories of compact objects in the stable homotopy category. Later, Hopkins [17] and Neeman [22, 23] transferred this idea to algebra, and were able to completely classify the localizing and colocalizing subcategories and thick subcategories of compact objects in the unbounded derived category $\mathsf{D}(A)$ over any commutative noetherian ring A. In all these cases, the space that governs this classification is the classical prime spectrum $\operatorname{Spec}(A)$.

More specifically, if A is a commutative noetherian ring, Neeman shows in [22, Theorem 2.8] that there is a bijection between localizing subcategories of $\mathsf{D}(A)$ and subsets of $\operatorname{Spec}(A)$. This is not just an abstract bijection, but instead it is based on the notion of the support of a cochain complex. A similar result, building on work of Hopkins, is shown in [22, Theorem 1.5], where Neeman classified all thick subcategories of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\operatorname{Proj}(A))$, the full triangulated subcategory of $\mathsf{D}(A)$ composed of all perfect complexes, which are exactly the compact objects in $\mathsf{D}(A)$. The dual result, classifying all colocalizing subcategories of $\mathsf{D}(A)$, and showing that it is also in bijection with the subsets of $\operatorname{Spec}(A)$, was shown by Neeman in [23, Theorem 0.2], answering a question of Chorny.

Studying the same question in the DG-setting, in [6, Theorem 8.1] and [7, Theorem 10.3], Benson, Iyengar and Krause were able to prove similar results about the derived category of a commutative DG-ring A, provided that $\mathrm{H}(A)$ is noetherian and that moreover A is formal. In that case, the space classifying these objects was $\operatorname{Spec}(\mathrm{H}(A))$. Here, being formal means that, up to quasi-isomorphism, the differential of A is equal to 0.

In our paper [P5] we were able to remove the formal assumption, and to show that similar classification results holds for any commutative DG-ring A, provided that $\mathrm{H}^0(A)$ is noetherian and that A has bounded cohomology. Our result showed that for such DG-rings A, the localizing and colocalizing subcategories and thick subcategories of compact objects in $\mathsf{D}(A)$ are in bijection with certain subsets of $\operatorname{Spec}(\mathrm{H}^0(A))$. This result provided a formal justification to a major principal in derived algebraic geometry, namely, that the space underlying a derived scheme is its underlying classical scheme. This implies that the functors

$$- \otimes^{\mathrm{L}}_A \mathrm{H}^0(A), \mathrm{R}\operatorname{Hom}_A(\mathrm{H}^0(A), -) : \mathsf{D}(A) \to \mathsf{D}(\mathrm{H}^0(A))$$

induce bijections between these global structures on $\mathsf{D}(A)$ and those on $\mathsf{D}(\mathrm{H}^0(A))$.

0.5 Finitistic dimensions

As discussed in Section 0.2, associated to a commutative ring A is its Krull dimension $\dim(A)$ which is a numerical invariant that correctly reflects the dimension of the geometric space associated to A.

In case A is a regular local ring, it was shown in [2, Theorem 4.7] that $\dim(A)$ is equal to the global dimension of A, which is defined by the formula

$$\operatorname{gl}\dim(A) = \sup_{M \in \operatorname{Mod}(A)} \operatorname{proj}\dim_A(M).$$

If a commutative noetherian local ring A is not regular, then its global dimension is always infinite. This is true even in very simple cases. For instance, if $A = \mathbb{K}[x]/(x^2)$, where $\mathbb{K}$ is a field, the ring of dual numbers, then as $\operatorname{proj}\dim_A(\mathbb{K}) = +\infty$, it follows that $\operatorname{gl}\dim(A) = +\infty$, despite the fact that A is a very simple ring, which has Krull dimension 0.

It turns out that even in such singular cases, there is still a way to approach the dimension of A using projective dimension. This is via the notions of finitistic dimensions, first defined by Auslander and Buchsbaum in [2]. If A is any ring (not necessarily commutative), the small (left) finitistic dimension of A is defined by the formula

$$\operatorname{fpd}(A) = \sup_{M \in \operatorname{Mod}_f(A), \operatorname{proj}\dim_A(M) < \infty} \operatorname{proj}\dim(A).$$

where $\operatorname{Mod}_f(A)$ denotes the category of finitely generated left A-modules. Similarly, the big finitistic dimension of A is given by

$$\operatorname{FPD}(A) = \sup_{M \in \operatorname{Mod}(A), \operatorname{proj}\dim_A(M) < \infty} \operatorname{proj}\dim(A).$$

It follows immediately that $\operatorname{fpd}(A) \leq \operatorname{FPD}(A)$. If A is a two-sided noetherian ring and $\operatorname{gl}\dim(A) < \infty$ then these three invariants coincide, so that

$$\operatorname{fpd}(A) = \operatorname{FPD}(A) = \operatorname{gl}\dim(A).$$

In case of the ring of dual numbers $A = \mathbb{K}[x]/(x^2)$, we saw that $\operatorname{gl}\dim(A) = +\infty$, but one can show that $\operatorname{fpd}(A) = \operatorname{FPD}(A) = 0$, reflecting the simplicity of $\operatorname{Mod}(A)$.

For commutative noetherian rings A, there is a very good understanding of the finitistic dimensions of A. It follows immediately from the Auslander-Buchsbaum formula ([2, Theorem 3.7]) that if A is a commutative noetherian local ring then $\operatorname{fpd}(A) = \operatorname{depth}(A)$.

As for the big finitistic dimension, Bass showed in [4, Corollary 5.5] that if A is a commutative noetherian ring then $\operatorname{FPD}(A) \geq \dim(A)$. He further mentioned that he does now know a ring which this $\operatorname{FPD}(A) < \dim(A)$. A decade later, in their deep and influential work [29], Raynaud and Gruson established the converse inequality $\operatorname{FPD}(A) \leq \dim(A)$, which implies that for any commutative noetherian ring A it holds that $\operatorname{FPD}(A) = \dim(A)$. This required them to prove the much more difficult result which states that if M is any A-module of finite projective dimension then $\operatorname{proj}\dim_A(M) \leq \dim(A)$.

If A is a commutative noetherian local ring, since its small finitistic dimension coincides with its depth, and its big finitistic dimension coincides with its Krull dimension, it follows that A is a Cohen-Macaulay ring if and only if

$$\operatorname{fpd}(A) = \operatorname{FPD}(A). \tag{CM6}$$

In [P6] we generalized the theory of finitistic dimensions to the setting of commutative noetherian DG-rings A with bounded cohomology. In particular, in [P6, Corollary 6.6] we give a characterization of local-Cohen-Macaulay DG-rings in term of their small finitistic dimensions. It is interesting to note however, that unlike the case of rings, the dimension does not determine uniquely the big finitistic dimension of such an A, and instead, as we show in [P6, Theorem 6.2] there is an inequality

$$\dim(\mathrm{H}^0(A)) - \operatorname{amp}(A) \leq \operatorname{FPD}(A) \leq \dim(\mathrm{H}^0(A)).$$

Moreover, this result is optimal, in the sense that there exist examples which achieve either bound.

The understanding of finitistic dimensions for noncommutative rings is much more limited. Let A be a (possibly noncommutative) finite dimensional algebra over a field $\mathbb{K}$, or more generally, an artin algebra. In that situation, three conjectures regarding the finitistic dimensions of A arose in the 1960s (though it is not clear by whom and when exactly):

1. $\operatorname{fpd}(A) = \operatorname{FPD}(A)$.
2. $\operatorname{fpd}(A) < \infty$.
3. $\operatorname{FPD}(A) < \infty$.

The first conjecture was disproved by Zimmermann-Huisgen in 1992 [41]. The other two remain wide open, and are known as the small and the big finitistic dimension conjectures. The paper [42] discusses the history of these problems. One reason these conjectures are so important is that they are known to imply many other homological conjectures. The introduction section of the paper [12] contains various important conjecture that the finitistic dimension conjecture implies.

A few years ago, Rickard developed in [30] a completely new approach for establishing that a finite dimensional algebra A over a field $\mathbb{K}$ satisfies that $\operatorname{FPD}(A) < \infty$. In general, if A is any ring, we denote by $\operatorname{Inj}(A)$ the set of all injective A-modules, and by $\operatorname{Loc}(\operatorname{Inj}(A))$ the smallest localizing subcategory of $\mathsf{D}(A)$ that contains $\operatorname{Inj}(A)$. One says that injectives generate $\mathsf{D}(A)$ if $\operatorname{Loc}(\operatorname{Inj}(A)) = \mathsf{D}(A)$. In [30, Theorem 4.3], Rickard shows that if A is a finite dimensional algebra over a field, for which injectives generate, then $\operatorname{FPD}(A) < \infty$. He also noted in [30, Theorem 3.3], that the Hopkins-Neeman theory, explained in Section 0.4 above, implies that injectives generate $\mathsf{D}(A)$ if A is a commutative noetherian ring.

Our papers [P7, P8] used DG-ring methods to obtain far reaching generalizations of the work of Rickard, and to obtain better understanding of the full scope of these ideas. A key concept used in these developments is the notion of a dualizing complex over a noncommutative ring. Over commutative rings, dualizing complexes were introduced by Grothendieck and discussed in Section 0.2 above. In the noncommutative case, this notion was first introduced by Yekutieli in [37]. For the purpose of this thesis, we use a slightly modified version of the definition of Yekutieli, following Iyengar-Krause [19]. If A is a two-sided noetherian ring, a dualizing complex over A, is a bounded complex of A-A-bimodules R, such that considered as a complex of either left or right

A-modules, it is made entirely of injective modules, such that its cohomologies are finitely generated both on the left and on the right, and such that the natural maps

$$A \to \mathrm{Hom}_A(R, R), \quad A \to \mathrm{Hom}_{A^{\mathrm{op}}}(R, R)$$

are both quasi-isomorphisms. The definition implies that if M is a finitely generated left A-module, then the natural map

$$M \to \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{R}\,\mathrm{Hom}_A(M, R), R)$$

is an isomorphism in $\mathsf{D}(A)$.

The class of rings that possess dualizing complexes contains all finite dimensional algebras over a field and all artin algebras. It is however much larger. For instance, in commutative algebra, virtually all commutative noetherian rings that arise in algebraic geometry possess dualizing complexes, but commutative artin algebras are all of Krull dimension 0. Thus, in some sense, generalizing results from artin algebras to two-sided noetherian rings with a dualizing complex is sort of generalizing from dimension 0 to an arbitrary positive dimension.

In [P7, Corollary 5.3] we use DG-ring methods to generalize Rickard's theorem, and prove that if A is a two-sided noetherian ring with a dualizing complex, for which injectives generate $\mathsf{D}(A)$, then $\mathrm{FPD}(A) < \infty$. We further observe that this result is optimal, because there exist noetherian rings for which injectives generate $\mathsf{D}(A)$ such that $\mathrm{FPD}(A) = +\infty$. These rings of course do not possess dualizing complexes.

Our later paper [P8] went further. We observed that the implication

$$\text{Injectives generate } \mathsf{D}(A) \implies \mathrm{FPD}(A) < \infty$$

for two-sided noetherian rings with a dualizing complex can be refined to

$$\text{Injectives generate } \mathsf{D}(A) \implies \mathsf{K}^+_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0 \iff \mathrm{FPD}(A) < \infty.$$

To explain the condition

$$\mathsf{K}^+_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0,$$

let us recall that one of the most basic facts in homological algebra is the uniqueness, up to homotopy equivalence, of projective resolutions of a given module. Since the zero module is projective, and since any bounded above acyclic complex of projective modules can be considered as a projective resolution of the zero module, it follows that any bounded above acyclic complex of projective modules is contractible.

Considering the homotopy category of complexes of projective modules $\mathsf{K}(\mathrm{Proj}(A))$, one may consider its full triangulated subcategory $\mathsf{K}_{\mathrm{ac}}(\mathrm{Proj}(A))$ which consists of all acyclic cochain complexes of projective modules. It follows from [19, Theorem 5.3] that one may view $\mathsf{K}_{\mathrm{ac}}(\mathrm{Proj}(A))$ as a kind of singularity category. In particular, if A is a two-sided noetherian ring and $\mathrm{gl}\dim(A) < \infty$, then $\mathsf{K}_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0$.

The basic homological fact that over any ring any **bounded above** acyclic complex of projective modules is contractible can then be stated as

$$\mathsf{K}^{-}_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0,$$

since $\mathsf{K}^{-}(\mathrm{Proj}(A))$ consists of all bounded above complexes of projectives. This also explains the meaning of the condition

$$\mathsf{K}^{+}_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0,$$

which says that any **bounded below** acyclic complex of projective modules is contractible. It turns out that for two-sided noetherian rings A with a dualizing complex, this condition is equivalent to $\mathrm{FPD}(A) < \infty$. Many implications of this fact are discussed in [P8, Section 5].

As far as we know, the paper [P8] was the first to explicitly discuss this question, but it is interesting to note that since then several other development occurred. In an appendix to [P8] written by Nakamura and Thompson, it is shown that this contractibility condition holds over any commutative noetherian ring. In particular, there exist noetherian rings for which this condition holds, but they have infinite finitistic dimension. Necessarily, they do not have dualizing complexes. Recently, in [27], Positelski gave several examples of rings over which there exist a bounded below acyclic complex of projectives which is not contractible. All his examples are over rings which are noncommutative and non-noetherian. Even more recently, in [10], Haesemeyer showed that over any commutative ring, any bounded below acyclic complex of finitely generated projective modules is contractible. Canonaco, Neeman and Stellari then used this result to deduce some basic structural results about derived categories of quasi-coherent sheaves over any quasi-compact, quasi-separated scheme.

Thus, it seems that the condition

$$\mathsf{K}^{+}_{\mathrm{ac}}(\mathrm{Proj}(A)) = 0,$$

which, as explained, is equivalent to the finitistic dimension conjecture, is intimately related to many exciting and deep questions in homological algebra.

0.6 Summary of contributions

We finish this introduction with a very brief summary of the main contributions of this thesis.

1. We resolved a basic question about Hochschild cohomology of adic algebras raised by Buchweitz and Flenner, and more generally, proved that in nice situations, Hochschild cohomology commutes with adic completion.

2. We developed the theory of Cohen-Macaulay DG-rings, generalized six different characterizations of Cohen-Macaulay rings to the differential graded setting, and proved that any derived quotient of a Cohen-Macaulay ring by a finite sequence of elements is a Cohen-Macaulay DG-ring.

3. We developed the notion of a sequence-regular commutative DG-ring, showed that this is the generic situation for derived algebraic varieties, and generalized the Auslander-Buchsbaum-Serre theorem about localization of regular local rings to the DG setting.

4. We have lifted the Hopkins-Neeman classification of localizing and colocalizing subcategories of the derived category from commutative noetherian rings to commutative noetherian DG-rings with bounded cohomology.

5. We developed the theory of finitistic dimension over commutative DG-rings, generalizing foundational results of Bass and Raynaud-Gruson from commutative noetherian rings to commutative noetherian DG-rings with bounded cohomology.

6. We generalized Rickard's theorem about generation of injectives implying finite finitistic dimension from finite dimensional algebras over a field to the class of all two-sided noetherian rings with a dualizing complex.

7. We introduced the condition that any bounded below acyclic cochain complex of projective modules is contractible, and proved that for two-sided noetherian rings with a dualizing complex, it is equivalent to having finite finitistic dimension.

Bibliography

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman. Local homology and cohomology on schemes. *Ann. Sci. École Norm. Sup. (4)*, 30(1):1–39, 1997. Correction: http://www.math.purdue.edu/ lipman.

[2] M. Auslander and D. A. Buchsbaum. Homological dimension in local rings. *Trans. Amer. Math. Soc.*, 85:390–405, 1957.

[3] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[4] H. Bass. Injective dimension in Noetherian rings. *Trans. Amer. Math. Soc.*, 102:18–29, 1962.

[5] H. Bass. On the ubiquity of Gorenstein rings. *Math. Z.*, 82:8–28, 1963.

[6] D. Benson, S. B. Iyengar, and H. Krause. Stratifying triangulated categories. *J. Topol.*, 4(3):641–666, 2011.

[7] D. J. Benson, S. B. Iyengar, and H. Krause. Colocalizing subcategories and cosupport. *J. Reine Angew. Math.*, 673:161–207, 2012.

[8] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[9] R.-O. Buchweitz and H. Flenner. Power series rings and projectivity. *Manuscripta Math.*, 119(1):107–114, 2006.

[10] A. Canonaco, C. Haesemeyer, A. Neeman, and P. Stellari. The passage among the subcategories of weakly approximable triangulated categories, 2024.

[11] H. Cartan. Sur les groupes d'Eilenberg-Mac Lane $H(\Pi, n)$. I. Méthode des constructions. *Proc. Nat. Acad. Sci. U.S.A.*, 40:467–471, 1954.

[12] E. L. Green, C. Psaroudakis, and O. y. Solberg. Reduction techniques for the finitistic dimension. *Trans. Amer. Math. Soc.*, 374(10):6839–6879, 2021.

[13] R. Hartshorne. *Residues and duality*, volume No. 20 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.

[14] R. Hartshorne. *Local cohomology*, volume No. 41 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967. A seminar given by A. Grothendieck, Harvard University, Fall, 1961.

[15] G. Hochschild. On the cohomology groups of an associative algebra. *Ann. of Math. (2)*, 46:58–67, 1945.

[16] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.

[17] M. J. Hopkins. Global methods in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 73–96. Cambridge Univ. Press, Cambridge, 1987.

[18] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[19] S. Iyengar and H. Krause. Acyclicity versus total acyclicity for complexes over Noetherian rings. *Doc. Math.*, 11:207–240, 2006.

[20] W. Krull. Beiträge zur arithmetik kommutativer integritätsbereiche. iii. zum dimensionsbegriff der idealtheorie. *Mathematische Zeitschrift*, 42:745–766, 1937.

[21] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[22] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[23] A. Neeman. Colocalizing subcategories of $D(R)$. *J. Reine Angew. Math.*, 653:221–243, 2011.

[24] T. Ogoma. Existence of dualizing complexes. *J. Math. Kyoto Univ.*, 24(1):27–48, 1984.

[25] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. *Inst. Hautes Études Sci. Publ. Math.*, 42:47–119, 1973.

[26] M. Porta, L. Shaul, and A. Yekutieli. On the homology of completion and torsion. *Algebr. Represent. Theory*, 17(1):31–67, 2014.

[27] L. Positselski. A bounded below, noncontractible, acyclic complex of projective modules, 2023.

[28] D. Puppe. On the formal structure of stable homotopy theory. Colloq. algebr. Topology, Aarhus 1962, 65-71 (1962)., 1962.

[29] M. Raynaud and L. Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Invent. Math.*, 13:1–89, 1971.

[30] J. Rickard. Unbounded derived categories and the finitistic dimension conjecture. *Adv. Math.*, 354:106735, 21, 2019.

[31] S. Schwede and B. Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[32] J.-P. Serre. Sur la dimension homologique des anneaux et des modules noethériens. In *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955*, pages 175–189. Science Council of Japan, Tokyo, 1956.

[33] L. Shaul. Injective DG-modules over non-positive DG-rings. *J. Algebra*, 515:102–156, 2018.

[34] L. Shaul. Completion and torsion over commutative DG rings. *Israel J. Math.*, 232(2):531–588, 2019.

[35] L. Shaul. Categorical properties of reduction functors over non-positive DG-rings. *Proc. Amer. Math. Soc.*, 152(2):489–500, 2024.

[36] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, 239:xii+253, 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[37] A. Yekutieli. Dualizing complexes over noncommutative graded algebras. *J. Algebra*, 153(1):41–84, 1992.

[38] A. Yekutieli. Duality and tilting for commutative DG-rings. arXiv:1312.6411v4, 2016.

[39] A. Yekutieli. Flatness and completion revisited. *Algebr. Represent. Theory*, 21(4):717–736, 2018.

[40] A. Yekutieli. *Derived categories*, volume 183 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.

[41] B. Zimmermann-Huisgen. Homological domino effects and the first finitistic dimension conjecture. *Invent. Math.*, 108(2):369–383, 1992.

[42] B. Zimmermann Huisgen. The finitistic dimension conjectures—a tale of 3.5 decades. In *Abelian groups and modules (Padova, 1994)*, volume 343 of *Math. Appl.*, pages 501–517. Kluwer Acad. Publ., Dordrecht, 1995.

Chapter 1

Hochschild cohomology commutes with adic completion

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