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HABILITATION THESIS

Legendrian submanifolds and exact Lagrangian cobordisms

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Contents

1. Introduction

Symplectic geometry is a branch of differential geometry that studies symplectic and contact manifolds and their Lagrangian and Legendrian submanifolds. It originated as a formalization of the mathematical apparatus of classical mechanics and geometric optics. Symplectic geometry radically changed after the 1985 article of Gromov [75] on pseudoholomorphic curves and the subsequent work of Floer [63] giving birth to symplectic topology or "Floer mehods" (sometimes called "hard methods") of symplectic geometry. Symplectic topology has emerged as an important field that has contributed to key developments in low-dimensional topology, geometry and mathematical physics.

A symplectic manifold (X, ω) consists of a smooth $2n$-dimensional manifold X and a closed non-degenerate 2-form ω. The canonical example of of a symplectic $2n$-manifold is $\mathbb{R}^{2n}$ with the coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and with the symplectic form $\sum_i dx_i \wedge dy_i$. One way to study a symplectic manifold is through its Lagrangian submanifolds, $L \subset (X, \omega)$ which are half-dimensional submanifolds, i.e. $\dim L = n$ with the property that $\omega|_{TL} = 0$.

Odd-dimensional analogues of symplectic manifolds are called contact manifolds, these are manifolds Y of dimension $2n-1$ equipped with a maximally non-integrable hyperplane field ξ called contact structure. Locally a contact structure is given by kernel of a 1-form α such that $\alpha \wedge d\alpha^{n-1} \neq 0$. Legendrian submanifolds of Y are $n-1$-dimensionall submanifolds such that $T_p\Lambda \subset \xi_p$ for all $p \in \Lambda$. The canonical example of a contact $2n-1$-manifold is $\mathbb{R}^{2n-1}$ with the coordinates $(x_1, \ldots x_{n-1}, y_1, \ldots, y_{n-1}, z)$ and with the standard contact structure given by $\ker(dz - \sum_i y_i dx_i)$, which we will denote by $\mathbb{R}^{2n-1}_{st}$.

Both symplectic and contact manifolds have no local invariants thanks to the corresponding Darboux theorems, but one can define global invariants using Lagrangian submanifolds, and Legendrian submanifolds, respectively. This approach to studying symplectic and contact manifolds is difficult to follow as Lagrangian submanifolds and Legendrian submanifolds are not classified in general. The geometries of contact and symplectic manifolds are closely related. For instance, the symplectic and contact conditions are compatible enough that one can consider symplectic manifolds whose boundary is a contact manifold. Weinstein domains (and more generally Weinstein cobordisms), certain symplectic manifolds with a handlebody decomposition that is compatible with the symplectic structure, are examples of symplectic manifolds with a contact boundary. Lagrangian cobordisms with Legendrian boundary are the relative version of closed Lagrangians. They are natural objects to consider in symplectic manifolds with contact boundary. In the case of Weinstein domains, their symplectic topology is encoded by Legendrian submanifolds. Therefore, studying Legendrian submanifolds and (exact) Lagrangian cobordisms between them provides a new perspective on symplectic manifolds with contact boundary. In this thesis we study Legendrian submanifolds and (exact) Lagrangian cobordisms in Weinstein domains and cobordisms.

2. Structure of thesis

The thesis represents the following papers:

- R. Golovko, *A note on Lagrangian cobordisms between Legendrian submanifolds of* $\mathbb{R}^{2n+1}$, Pacific J. Math., 261(1) (2013) 101–116.
- R. Golovko, *A note on the front spinning construction*, Bulletin of the London Mathematical Society, 46 (2014), no. 2, 258–268.
- B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko, *Floer homology and Lagrangian concordance*, Proceedings of the Gökova Geometry/Topology Conference 2014, (2015) 76–113.
- G. Dimitroglou Rizell and R. Golovko, *Estimating the number of Reeb chords using a linear representation of the characteristic algebra*, Algebraic and Geometric Topology, 15-5 (2015), 2887–2920.
- B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini and R. Golovko, *Noncommutative augmentation categories*, Proceedings of the Geometry-Topology Conference 2015. Gökova Geometry/Topology Conference (GGT), Gökova, 2016, 116–150.
- B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini and R. Golovko, *Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors*, available at arXiv:1712.09126, accepted for publication in Annales Scientifiques de l'École Normale Supérieure.
- G. Dimitroglou Rizell and R. Golovko, *The stable Morse number as a lower bound for the number of Reeb chords*, Journal of Symplectic Geometry, Vol. 16, No. 5 (2018), 1209-1248.
- B. Chantraine, G. Dimitroglou Rizell, P. Ghiggini, and R. Golovko, *Floer theory for Lagrangian cobordisms*, Journal of Differential Geometry, Volume 114, Number 3 (2020), 393-465.
- G. Dimitroglou Rizell and R. Golovko, *On Legendrian products and twist spuns*, Algebr. Geom. Topol., 21-2 (2021), 665–695.
- R. Golovko, *On topologically distinct infinite families of exact Lagrangian fillings*, Archivum Mathematicum, Vol. 58 (2022), 287–293.
- R. Golovko, *A note on infinite number of exact Lagrangian fillings for spherical spuns*, Pacific Journal of Mathematics 317-1 (2022), 143–152.
- R. Golovko, *On torsion in linearized Legendrian contact homology*, Journal of Knot Theory and Its Ramifications, Vol. 32, No. 07, 2350056 (2023).
- R. Golovko, *On non-geometric augmentations in high dimensions*, Geometriae Dedicata, 217, 104 (2023).

In Chapter 3 we discuss the basic definitions of the theory of Legendrian submanifolds. After that, in Chapters 4–12 we discuss the results presented in the mentioned above papers.

Acknowledgement

We would like to thank Baptiste Chantraine, Georgios Dimitroglou Rizell and Paolo Ghiggini for the fruitful and enjoyable discussions and for collaboration on parts of the material included in this thesis.

3. Invariants of Legendrian submanifolds of $\mathbb{R}^{2n+1}$

Given a contact manifold (X, ξ), where $\xi = \ker \alpha$, we define the *Reeb vector field* R_α to be given by the equations $\alpha(R_\alpha) = 1$, $i_{R_\alpha}(\alpha) = 0$.

A *Liouville domain* is a compact symplectic manifold $(W, d\theta)$ with a choice of primitive of the symplectic form such that the *Liouville vector field* defined by $i_{\mathcal{L}} d\theta = \theta$ is transverse to ∂W and points outwords. Then, given a Liouville domain $(W, d\theta, \mathcal{L})$, there is a standard way of completing it: the condition that $\mathcal{L}$ is outward transverse to the boundary implies that there is a symplectic collar neighborhood $\partial W \times (-\varepsilon, 0]$ with

$$\theta|_{\partial W \times \{t\}} = e^t \theta|_{\partial W \times \{0\}}.$$

Therefore, we can attach a cylindrical end $\partial W \times [0, \infty)$ with the primitive of sympectic form given by the same formula $\theta|_{\partial W \times \{t\}} = e^t \theta|_{\partial W \times \{0\}}$ to get a non-compact manifold $P = W \cup (\partial W \times [0, \infty))$ with complete Liouville vector field. Such a manifold is called a *Liouville manifold*.

In this thesis, the contact manifolds that we will use are the standard contact vector space $\mathbb{R}^{2n+1}_{st}$ and the so-called *contactisation* of a Liouville manifold $(P, \omega = d\theta)$ that is defined to be $(P \times \mathbb{R}, dz + \theta)$, where z is the coordinate on the first $\mathbb{R}$-factor. It is easy to see that for both $\mathbb{R}^{2n+1}_{st}$ and $(P \times \mathbb{R}, dz + \theta)$ the Reeb vector field is given by ∂_z.

From now on in this section we will discuss only the case of $\mathbb{R}^{2n+1}_{st}$ that we will often write simply as $\mathbb{R}^{2n+1}$. Note that most of the discussion in this section and invariants that we define here naturally extend to the situation of $(P \times \mathbb{R}, dz + \theta)$.

3.1 Legendrian submanifolds

An immersion of an n-manifold into $\mathbb{R}^{2n+1}$ is *Legendrian* if it is everywhere tangent to the hyperplane field ξ, and the image of a Legendrian embedding is a *Legendrian submanifold*. The *Reeb chords* of a Legendrian submanifold Λ are segments of flow lines of R_α starting and ending at points of Λ. The *symplectisation* of $\mathbb{R}^{2n+1}$ is the exact symplectic manifold $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha))$, where t is a coordinate on $\mathbb{R}$. There are two natural projections, the first one is called the *front projection* and it is defined by

$$\Pi_F(x_1, y_1, \ldots, x_n, y_n, z) = (x_1, \ldots, x_n, z)$$

and the second one which is called the *Lagrangian projection* and it is given by

$$\Pi_L(x_1, y_1, \ldots, x_n, y_n, z) = (x_1, y_1, \ldots, x_n, y_n)$$

The Lagrangian projection $\Pi_L(\Lambda)$ of a Legendrian submanifold Λ is an exact Lagrangian immersion into $\mathbb{R}^{2n}$. Note that for closed Legendrians generically the self-intersection of $\Pi_L(\Lambda)$ consists of a finite number of transverse double points. These points correspond to the set of Reeb chords of Λ. The set of Reeb chrods is typically denoted by $\mathcal{Q}(\Lambda)$. A Legendrian submanifold is called *chord generic* if it has a finite number of Reeb chords. From now on we assume that all Legendrian submanifolds of $\mathbb{R}^{2n+1}$ are connected and chord generic.

3.2 Classical and non-classical invariants of Legendrian submanifolds

There are two classical invariants of a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$, namely the *Thurston–Bennequin invariant* (number) and the *rotation class*.

The Thurston–Bennequin invariant has been defined by Bennequin in [9], and independently by Thurston for Legendrian knots in $\mathbb{R}^3$. Then Tabachnikov generalized it to higher dimensions, see [101]. The Thurston–Bennequin number $tb(\Lambda)$ of a closed, oriented Legendrian $\Lambda \subset \mathbb{R}^{2n+1}$ isdefined to be the linking number $lk(\Lambda, \Lambda')$. Here Λ' is an oriented submanifold which is obtained from Λ by a small shift in the Reeb direction.

The rotation class $r(\Lambda)$ for a Legendrian submanifold of $\mathbb{R}^{2n+1}$ has been defined by Ekholm, Etnyre and Sullivan [46]. It is equal to the homotopy class of $(f, df_{\mathbb{C}})$ in the space of complex fiberwise isomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$, where $f : \Lambda \to \mathbb{R}^{2n+1}$ is an embedding of Λ. In the situation when $\Lambda = S^n$ and n is odd, $r(\Lambda) \in \pi_n(U(n)) \simeq \mathbb{Z}$ and it is called $r(\Lambda)$ the *rotation number*.

Legendrian contact homology is a modern invariant of a closed Legendrian submanifold of $\mathbb{R}^{2n+1}$. For Legendrians in $\mathbb{R}^3$ it has appreared in [26], and then in [47] it has been extended to $\mathbb{R}^{2n+1}$ for all $n \geq 1$; finally, the case of $P \times \mathbb{R}$ has been described in [49]. The Legendrian contact homology of a closed, orientable Legendrian submanifold Λ with the finite set of Reeb chords $\mathcal{Q}(\Lambda)$ is the homology of the noncommutative differential graded algebra $(\mathcal{A}_\Lambda, \partial_\Lambda)$ over a unital ring R freely generated by the elements of $\mathcal{Q}(\Lambda)$ and is denoted by $LCH_*(\Lambda)$. The differential of $\mathcal{A}_\Lambda$ counts certain holomorphic curves in the symplectisation of $\mathbb{R}^{2n+1}$, whose domains are unit disks with the boundary with points removed from the boundary. At these points, the holomorphic curve has one positive asymptotic and several negative asymptotics. For the details we refer the reader to [47].

Note that $(\mathcal{A}_\Lambda, \partial_\Lambda)$ and even its homology are difficult to compute, and hence are difficult to use for practical applications. One way to deal with this issue and to extract useful information from $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is to follow the Chekanov's method of linearization. For that we need the notion of augmentation. An augmentation ε is an algebra homomorphism from $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to $(R, 0)$ which satisfies $\varepsilon(1) = 1$ and $\varepsilon \circ \partial_\Lambda = 0$. Augmentation ε allows us to linearize the differential graded algebra to a complex $LC^\varepsilon := (A_\Lambda, \partial^\varepsilon_{1,\Lambda})$ with homology groups $LCH^\varepsilon_*(\Lambda)$. Here A_Λ is the free R-module generated by the elements of $\mathcal{Q}(\Lambda)$. We define $LCH^*_\varepsilon(\Lambda)$ to be the homology of the dual complex $LC_\varepsilon(\Lambda) := Hom(LC^\varepsilon(\Lambda), R)$. The linearized homology (cohomology) groups may depend on the choice of ε. However, the set of graded groups $\{LCH^\varepsilon_*(\Lambda)\}$ ($\{LCH^*_\varepsilon(\Lambda)\}$), where ε is any augmentation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, provides a Legendrian isotopy invariant, see [26, 47]. A similar construction, where one relies on two augmentations instead of one leads to the bilinearised Legendrian homology (cohomology), it has been defined by Bourgeois–Chantraine in [13], and is denoted by $LCH^{\varepsilon_1,\varepsilon_2}_*(\Lambda)$ ($LCH^*_{\varepsilon_1,\varepsilon_2}(\Lambda)$) for the basic properties we refer the reader to [13].

In [46], Ekholm, Etnyre and Sullivan used Legendrian contact homology to prove that for any $n > 1$ there is an infinite family of Legendrian embeddings of the n-sphere into $\mathbb{R}^{2n+1}$ that are not Legendrian isotopic even though they have the same classical invariants. They also prove similar results for Legendrian surfaces and n-tori, see [46]. These results indicate that the theory of Legendrian submanifolds of standard contact $(2n+1)$-space is very rich.

4. Exact Lagrangian cobordisms and classical invariants of Legendrian submanifolds

Lagrangian cobordism provides a relation between Legendrian submanifolds, and it is a crucial ingredient in the definition of the functorial properties of modern Legendrian invariants such as Legendrian contact homology in the spirit of the symplectic field theory (SFT) of Eliashberg, Givental and Hofer [56].

Consider a Liouville manifold $(P, d\theta)$, its contactisation $(Y = P \times \mathbb{R}, \alpha = dz + \theta)$ and its symplectisation $(\mathbb{R} \times Y, d(e^t\alpha))$, where t is a cooredinate on $\mathbb{R}$.

Let Λ^- and Λ^+ be two closed Legendrian submanifolds of (Y, α). An *exact Lagrangian cobordism from* Λ^- *to* Λ^+ in $(\mathbb{R} \times Y, d(e^t\alpha))$ is a properly embedded submanifold $L \subset \mathbb{R} \times Y$ without boundary satisfying the following conditions:

1. There exists $T \gg 0$ such that $L \cap ((-\infty, -T) \times Y) = (-\infty, -T) \times \Lambda^-$, $L \cap ((T, +\infty) \times Y) = (T, +\infty) \times \Lambda^+$, and $L \cap ([-T, T] \times Y)$ is compact.

2. There exists a smooth function $f_L : \Sigma \to \mathbb{R}$ for which $e^t\alpha|_L = df_L$, $f_L|_{(-\infty,-T)\times\Lambda^-}$ is constant, and $f_L|_{(T,\infty)\times\Lambda^+}$ is constant.

$(T, +\infty) \times \Lambda_+ \subset L$ and $(-\infty, -T) \times \Lambda_- \subset L$ are called the *positive end* and the *negative end* of L, respectively. A cobordism from a submanifold to itself is called *endocobordism*.

Example. If Λ is a closed Legendrian submanifold of (Y, ξ), then $\mathbb{R} \times \Lambda$ is an exact Lagrangian cobordism inside $(\mathbb{R} \times Y, d(e^t\alpha))$ from Λ to itself. Cobordisms of this type are called *trivial Lagrangian cylinders.*

In the case when there exists an exact Lagrangian cobordism L from Λ^- to Λ^+ we say that Λ^- *is exact Lagrangian cobordant to* Λ^+ and write $\Lambda^- \prec_L^{ex} \Lambda^+$. If Σ is an exact Lagrangian cobordism from the empty set to Λ, we call Σ an *exact Lagrangian filling* of Λ. In the latter case we say that Λ is *exactly fillable.*

Given exact Lagrangian cobordisms L_a from Λ^- to Λ and L_b from Λ to Λ^+, their *concatenation* $L_a \odot L_b$ is defined as follows.

First, translate L_a and L_b so that

$$\begin{aligned} L_a \cap ((-1, +\infty) \times Y) &= (-1, +\infty) \times \Lambda, \\ L_b \cap ((-\infty, 1) \times Y) &= (-\infty, 1) \times \Lambda. \end{aligned}$$

Then we define

$$L_a \odot L_b := (L_a \cap ((-\infty, 0] \times Y)) \cup (L_b \cap ([0, +\infty) \times Y)).$$

Since different choices of translation lead to Hamiltonian isotopic exact Lagrangian cobordisms, the following Lemma holds:

Lemma 1. The compactly supported Hamiltonian isotopy class of $L_a \odot L_b$ is independent of the above choices of translations.

The question about obstructions to the existence of Lagrangian cobordisms was first approached by Chantraine [19]: employing an adjunction inequality obtained through gauge theory, he showed that classical invariants (rotation number and Thurston-Bennequin invariant) of Legendrian knots can provide obstructions to the existence of Lagrangian cobordisms.

We prove the following high-dimensional version of Chantraine's result:

Theorem 2 ([69])**.** *Let Λ^- and Λ^+ be two closed, orientable, connected Legendrian submanifolds of $\mathbb{R}^{2n+1}$.*

(1) *If n is even and L is a smooth cobordisms from Λ^- to Λ^+, then*

$$tb(\Lambda_+) + tb(\Lambda^-) = (-1)^{\frac{n}{2}+1}\chi(L).$$

(2) *If n is odd, Λ_- is fillable and L is an exact Lagrangian cobordism from Λ^- to Λ^+, then*

$$tb(\Lambda)^+) - tb(\Lambda^-) = (-1)^{\frac{(n-2)(n-1)}{2}+1}\chi(L).$$

Endocobordisms

Rigidity of Lagrangian submanifolds has been discovered in many situations.

Dimitroglou Rizell and I proved the following rigidity result for exact Lagrangian endocobordisms of spin fillable Legendrian submanifolds of $P \times \mathbb{R}$:

Theorem 3 ([37])**.** *Let Λ be a spin fillable Legendrian submanifold of $P \times \mathbb{R}$, L be an exact Lagrangian endocobordism of Λ inside the symplectization, and $\mathbb{F}$ be a field. In addition, assume that $L_\Lambda * L$ is spin for any spin exact Lagrangian filling L_Λ of Λ. Then*

(1) $\dim H_i(L;\mathbb{F}) = \dim H_i(\Lambda;\mathbb{F})$ *for all i,*

(2) the map

$$(i^-_*, i^+_*) : H_j(\Lambda;\mathbb{F}) \to H_j(L;\mathbb{F}) \oplus H_j(L;\mathbb{F})$$

is injective for all j.

Here i^+ is the inclusion of Λ as a positive end of L, i^- is the inclusion of Λ as a negative end of L, and $\mathbb{F}$ is an arbitrary field. If $\mathbb{F} = \mathbb{Z}_2$, then the spin assumptions above can be omitted.

Moreover, when Λ is a homology sphere, Dimitroglou Rizell and I proved the following theorem:

Theorem 4 ([37])**.** *1. If Λ is a $\mathbb{Z}_2$-homology sphere admitting an exact Lagrangian filling, any exact Lagrangian endocobordism L of Λ has the property that the maps in homology*

$$i^{\pm}_*:\ H_j(\Lambda;\mathbb{Z}_2) \to H_j(L;\mathbb{Z}_2)$$

induced by the inclusions of the boundary are isomorphisms. In particular, L is spin and has vanishing Maslov class.

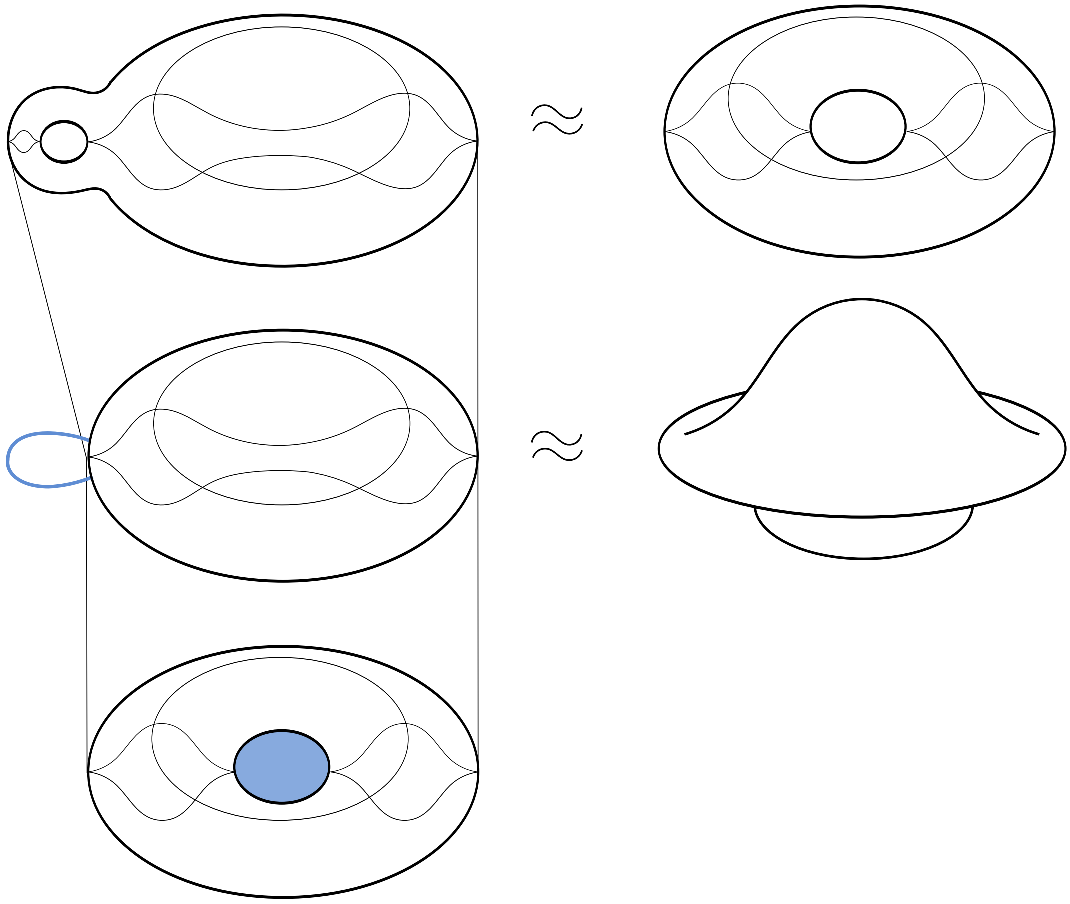

Figure 4.1: The exact Lagrangiann cobordism not diffeomorphic to the cylinder.

2. *If Λ is a $\mathbb{Z}$-homology sphere admitting an exact Lagrangian spin filling L_Λ, any exact Lagrangian endocobordism L of Λ satisfies the property that the maps in homology*

$$i^{\pm}_{*}\colon\ H_j(\Lambda;\mathbb{Z}) \to H_j(L;\mathbb{Z})$$

induced by the inclusions of the boundary are isomorphisms.

It is natural to ask whether every exact Lagrangian cobordism described in Theorem 3 is diffeomorphic to $\mathbb{R}\times\Lambda$. Dimitroglou Rizell and I in [37] constructed an example of an exact Lagrangian endocobordism of a fillable Legendrian $T^2 \subset \mathbb{R}^5$ which is not diffeomorphic to $\mathbb{R}\times T^2$, see Figure 4.1.

Murphy in [84] prove an h-principle type of result for a class of certain Legendrian embeddings in a contact manifolds of dimension at least 5. These Legendrian embeddings are called loose, see [84] for the definition and for the statement of the h-principle result in these settings.

Given an immersed exact Lagrangian cobordism with loose negative end, Eliashberg and Murphy in [58] provided the homotopy-theoretic conditions which determine when the double points can be removed, thus producing an embedded exact Lagrangian cobordism. Using this result, in contrast to Theorem 3, Dimitroglou Rizell and I prove the following theorems:

Theorem 5 ([37]). *For any closed, orientable, connected 3-manifold M and any closed, connected loose Legendrian surface $\Sigma_g \subset \mathbb{R}^5$ of genus g, there is an exact*

Lagrangian endocobordism L of Σ_g in the symplectization of $\mathbb{R}^5$ which is diffeomorphic to $M\#(\mathbb{R} \times \Sigma_g)$.

Theorem 6 ([37])**.** *For any loose Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$ and number $N > 0$ there exists an exact Lagrangian endocobordism L of Λ satisfying $\sum_i \dim H_i(L;\mathbb{F}) \geq N$. Here $\mathbb{F}$ is an arbitrary field.*

4.1 Lagrangian concordances

An exact Lagrangian concordance is a special case of an exact Lagrangian cobordism which is diffeomorphic to $\mathbb{R} \times \Lambda$ for some Legendrian Λ. In a joint work with Chantraine, Dimitroglou Rizell and Ghiggini, we classify exact Lagrangian concordances from the $tb = -1$ Legendrian unknot to itself in the tight contact three sphere.

Theorem 7 ([21])**.** *Let $\Lambda_0 \subset S^3$ be a Legendrian unknot with $tb(\Lambda_0) = -1$, and let L be an exact Lagrangian cobordism from Λ to itself inside the symplectisation of S^3. It follows that L is compactly supported Hamiltonian isotopic to $\phi^m(\mathbb{R}\times\Lambda_0)$ for some $m \in \mathbb{Z}$.*

Chantraine in [20] has shown that in dimension three the relation of being Lagrangian concordant is not symmetric, in particular there are Lagrangian concordances that cannot be inverted. Other examples in dimension three have later been constructed in [8] and [31]. In a joint work with Chantraine, Dimitroglou Rizell and Ghiggini, we generalize the construction of Chantraine to high dimensions and prove the following theorem.

Theorem 8 ([21])**.** *For every $m_1, \dots, m_k \in \mathbb{N}$ there exist fillable Legendrian submanifolds $\Lambda_1, \Lambda_2 \subset \mathbb{R}^{2(1+\sum_i m_i)+1}$ diffeomorphic to $S^1 \times S^{m_1} \times \cdots \times S^{m_k}$ with the property that*

- *there is an exact Lagrangian concordance from Λ_1 to Λ_2;*
- *there is no Lagrangian concordance from Λ_2 to Λ_1.*

One of the consequences of Theorem 8 is the following statement:

Corollary ([21]). For any $n \geq 2$, there exist two Legendrian submanifolds $\Lambda_1, \Lambda_2 \subset \mathbb{C}^n \times \mathbb{R}$ that are not loose and such that

- Λ_1 is not Legendrian isotopic to Λ_2, but
- there is an exact Lagrangian concordance from Λ_1 to Λ_2.

5. Floer theory for Lagrangian cobordisms and its applications

In order to study the topology of exact Lagrangian cobordisms and the properties of symplectic manifolds which contain these cobordisms (Weinstein cobordisms, Liouville cobordisms and so on), Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini and I introduce a version of Lagrangian Floer homology for pairs of exact Lagrangian cobordisms. This construction finds its inspiration in the work of Ekholm in [43], which gives a symplectic field theory point of view on wrapped Floer homology of Abouzaid and Seidel from [2].

The definition of this new Floer theory requires the use of augmentations of the Chekanov-Eliashberg algebras of the negative ends as bounding cochains in order to algebraically cancel certain degenerations of the holomorphic curves at the negative ends of the cobordisms. Bounding cochains have been introduced, in the closed case, by Fukaya, Oh, Ohta and Ono in [65], while augmentations, which play a similar role in the context of Legendrian contact homology, have been introduced by Chekanov in [26].

For a pair of exact Lagrangian cobordisms obtained by a suitable small Hamiltonian push-off, the construction that Chantraine, Dimitroglou Rizell, Ghiggini and I propose leads to several long exact sequences relating the singular homology of the cobordism with the Legendrian contact homology of its ends. These long exact sequences are used to give restrictions on the topology of exact Lagrangian cobordisms under various hypotheses on the topology of the Legendrian ends. Similar long exact sequences have been previously shown by Sabloff and Traynor, see [93], in the setting of generating family homology under the additional assumption that the cobordism admits a compatible generating family, and by the author in [69] in the case when the negative end of the cobordism admits an exact Lagrangian filling. Our results have been placed in a more general framework by Cieliebak and Oancea [28].

For the extra details on Legendrian contact homology necessary for the results in this section, we refer to [26, 50, 49, 62].

The notion of Lagrangian cobordism that we discuss is different from the notion of Lagrangian cobordisms introduced by Arnol'd in [6] and studied by Biran and Cornea in [10, 11]. Despite the differences, for Lagrangian cobordisms between Legendrian submanifolds with no Reeb chords, some of the obtained results resemble some of the results of Biran–Cornea [10, 11] and Suárez [99].

5.1 Main results.

Let $(P, d\theta)$ be a Liouville manifold and $(Y, \alpha) := (P \times \mathbb{R}, dz + \theta)$ its contactisation. We consider a pair of exact Lagrangian embeddings $\Sigma_0, \Sigma_1 \hookrightarrow X$, where $(X, \omega) = (\mathbb{R} \times Y, d(e^t\alpha))$ is the symplectisation of (Y, α). We assume that the positive and negative ends of Σ_i $i = 0, 1$ are cylindrical over Legendrian submanifolds Λ_i^- and Λ_i^+ respectively, and thus Σ_i is a Lagrangian cobordisms from Λ_i^- to Λ_i^+; see Figure 5.1 for a schematic representation. We assume that Σ_0 and Σ_1 intersect transversely and that their Legendrian ends are chord-generic in the sense of [49].

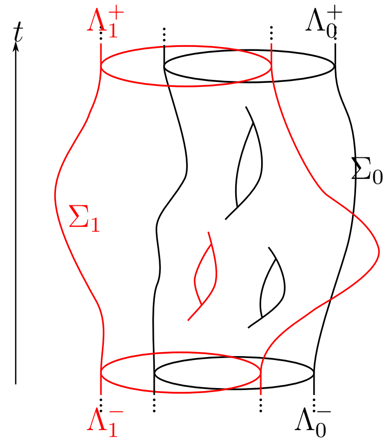

Figure 5.1: Two Lagrangian cobordisms inside a symplectisation $\mathbb{R} \times Y$, where the vertical axis corresponds to the $\mathbb{R}$-coordinate.

Let R be a ring of characteristic 2 or, if all Σ_i's and $\Lambda_i^{\pm}$'s are relatively Pin, any ring. We assume that $\mathcal{A}(\Lambda_i^-; R)$ of Λ_i^- admits an augmentation ε_i^- over R for $i = 0, 1$. It follows from the results of Ekholm, Honda and Kálmán in [52] that $\mathcal{A}(\Lambda_i^+; R)$ also admits an augmentation $\varepsilon_i^+ = \varepsilon_i^- \circ \Phi_{\Sigma_i}$, where $\Phi_{\Sigma_i} \colon \mathcal{A}(\Lambda_i^+; R) \to \mathcal{A}(\Lambda_i^-; R)$ is the unital DGA morphism induced by the cobordism Σ_i. Thus the bilinearised contact cohomologies $LCH_{\varepsilon_0^{\pm}, \varepsilon_1^{\pm}}(\Lambda_0^{\pm}, \Lambda_1^{\pm})$ are defined.

The *Cthulhu complex* of the pair (Σ_0, Σ_1) is the complex whose underlying graded R-module is

$$\mathrm{Cth}_\bullet(\Sigma_0, \Sigma_1) := C^\bullet(\Lambda_0^+, \Lambda_1^+)[2] \oplus C^\bullet(\Sigma_0, \Sigma_1) \oplus C^\bullet(\Lambda_0^-, \Lambda_1^-)[1]$$

for a unital ring R. Here $C^\bullet(\Lambda_0^{\pm}, \Lambda_1^{\pm})$ is the free graded R-module generated by the Reeb chords from $\Lambda_1^{\pm}$ to $\Lambda_0^{\pm}$ and $C^\bullet(\Sigma_0, \Sigma_1)$ is the free graded R-module generated by the intersection points $\Sigma_0 \cap \Sigma_1$.

Now we define the *Cthulhu differential* $\mathfrak{d}_{\varepsilon_0^-, \varepsilon_1^-}$, which is a differential of degree 1 on $\mathrm{Cth}_\bullet(\Sigma_0, \Sigma_1)$. With respect to the direct sum decomposition above, it has the form

$$\mathfrak{d}_{\varepsilon_0^-, \varepsilon_1^-} = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & d_{-0} & d_{--} \end{pmatrix}.$$

Loosely speaking, every non-zero entry in the presentation of $\mathfrak{d}_{\varepsilon_0^-, \varepsilon_1^-}$ is defined by counting rigid punctured pseudoholomorphic strips in the corresponding moduli spaces described in [24, Section 3.2], where the counts are weighted by the corresponding augmentations.

The Cthulhu complex turns out to be acyclic, i.e. the homology of it vanishes, because of its invariance properties with respect to a large class of Hamiltonian deformations. These Hamiltonian deformations in the contactisation of a Liouville manifold allow us to displace any pair of Lagrangian cobordisms.

When the negative ends are empty, this complex recovers the wrapped Floer cohomology complex as it appears in the work of Ekholm, see [43]. When the positive ends are empty and there are no homotopically trivial Reeb chords of both Λ_i^-'s, this complex is quite similar to the Floer complex presented by Akaho in [3, Section 8].

In order to deal with the ring R of characteristic different from two, which is essential for our applications, we define coherent orientations for the relevant moduli spaces of pseudoholomorphic curves. This can be done in the case when the Lagrangian cobordisms are relatively Pin (following Ekholm, Etnyre and Sullivan in [48] and Seidel in [95, Section 11]). The treatment of signs is done in the work of Karlsson [78].

5.2 Long exact sequences for LCH induced by a Lagrangian cobordism.

If Σ_1 is a Hamiltonian deformation of Σ_0 for some suitable and sufficiently small Hamiltonian, there is a well defined Floer differential on $CF(\Sigma_0, \Sigma_1)$ and the Floer homology group $HF(\Sigma_0, \Sigma_1)$ can be identified with the Morse homology group of Σ_0. Similarly, the bilinearised Legendrian contact homology groups $LCH_{\varepsilon_0^\pm, \varepsilon_1^\pm}(\Lambda_0^\pm, \Lambda_1^\pm)$ can be identified with the bilinearised contact homology groups $LCH_{\varepsilon_0^\pm, \varepsilon_1^\pm}(\Lambda_0^\pm)$ (as defined in [13]) following [45]. Moreover, in the same situation, the Cthulhu complex can be interpreted as a double cone, and thus provides long exact sequences which can be reinterpreted, by the identifications discussed above, as exact sequences relating the singular homology of a Lagrangian cobordism and the Legendrian contact homology of its ends.

In the rest of this section, Λ^+ and Λ^- will always denote closed Legendrian submanifolds of dimension n. We will denote by $\overline{\Sigma}$ the natural compactification of Σ obtained by adjoining its Legendrian ends $\Lambda_\pm$. Note that $\overline{\Sigma}$ is diffeomorphic to $\Sigma \cap [-T, +T] \times Y$ for some $T \gg 0$ sufficiently large. We will also use the notation $\partial_\pm \overline{\Sigma} := \Lambda_\pm \subset \overline{\Sigma}$, which implies that $\partial\overline{\Sigma} = \partial_+\overline{\Sigma} \sqcup \partial_-\overline{\Sigma}$.

5.2.1 A generalisation of the long exact sequence of a pair.

The first exact sequence we produce from a Lagrangian cobordism is given by the following:

Theorem 9 ([24]). *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. There is a long exact sequence*

$$\begin{array}{l} \cdots \longrightarrow LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \\ \qquad\qquad\downarrow \\ H_{n+1-k}(\overline{\Sigma}, \partial_-\overline{\Sigma}; R) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to \cdots, \end{array} \tag{5.1}$$

where the map $\Phi_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$ is the adjoint of the bilinearised DGA morphism Φ_Σ induced by Σ (see [52]).

When $\Lambda^- = \emptyset$ is empty the ong exact sequence above becomes the isomorphism

$$LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \xrightarrow{\cong} H_{n+1-k}(\overline{\Sigma}; R)$$

which appears in the work of Ekholm in [43]. Its proof was completed by Dimitroglou Rizell in [36]; also see [21] for an analogous isomorphism induced by a *pair* of fillings.

5.2.2 A generalisation of the duality long exact sequence and fundamental class

Theorem 10 ([24])**.** *Let Σ be an exact graded Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. Assume that Λ^- is horizontally displaceable; then there is a long exact sequence*

$$\begin{array}{l} \cdots \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to LCH^{\varepsilon_0^-,\varepsilon_1^-}_{n-k-1}(\Lambda^-) \to H_{n-k-1}(\Sigma; R) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad LCH^{k+1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \to \cdots \end{array} \tag{5.2}$$

In the previous theorem we use the notion of horizontally displaceble Legendrian submanifold. A Legendrian submanifold Λ is called *horizontally displaceable* if there exists a Hamiltonian isotopy ϕ_t of $(P, d\theta)$ which displaces the Lagrangian projection of Λ from itself.

If $\Sigma = \mathbb{R} \times \Lambda$, then $H_\bullet(\Sigma) = H_\bullet(\Lambda)$, and hence the above long exact sequence recovers the duality long exact sequence for Legendrian contact homology originally constructed by Sabloff in [92] for Legendrian knots and then generalised to arbitrary Legendrian submanifolds by Ekholm, Etnyre and Sabloff in [45]. In the bilinearised setting, the duality long exact sequence was introduced by Bourgeois and Chantraine in [13]. We use Exact Sequence (5.2) in [24] is used to prove that the fundamental class in LCH defined by Sabloff in [92] and Ekholm, Etnyre and Sabloff in [45] is functorial with respect to the maps induced by exact Lagrangian cobordisms.

5.2.3 A generalisation of the Mayer-Vietoris long exact sequence

Finally, we prove a version of the Mayer-Vietoris long exact sequence.

Theorem 11 ([24])**.** *Let Σ be an exact graded Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. Then there is a long exact sequence*

$$\begin{array}{l} \cdots \longrightarrow LCH^{k-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \\ \swarrow \\ H_{n-k}(\partial_-\overline{\Sigma}; R) \to LCH^k_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \oplus H_{n-k}(\overline{\Sigma}; R) \to LCH^k_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+) \\ \qquad\qquad\qquad\qquad\qquad \swarrow \\ \qquad\qquad\qquad\qquad \cdots \end{array} \tag{5.3}$$

where the component

$$H_{n-k}(\partial_-\overline{\Sigma}; R) \to H_{n-k}(\overline{\Sigma}; R)$$

of the left map is induced by the topological inclusion of the negative end.

If $\varepsilon_0^- = \varepsilon_1^- = \varepsilon$*, it moreover follows that the image of the fundamental class under the component* $H_n(\partial_-\overline{\Sigma}; R) \to LCH^0_{\varepsilon,\varepsilon}(\Lambda^-)$ *of the above morphism vanishes. Moreover, under the additional assumption that* Λ^- *is horizontally displaceable, the image of a generator under* $H_0(\partial_-\overline{\Sigma}; R) \to LCH^n_{\varepsilon,\varepsilon}(\Lambda^-)$ *is equal to the fundamental class in Legendrian contact homology.*

In particular we get that the fundamental class in $H_n(\partial_-\overline{\Sigma}; R)$ either is nonzero in $H_n(\overline{\Sigma})$, or is the image of a class in $LCH^{-1}_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$. In both cases, $\Lambda^+ \neq \emptyset$. Thus we obtain a new proof of the following result.

Corollary ([24, 34]). If $\Lambda \subset P \times \mathbb{R}$ admits an augmentation, then there is no exact Lagrangian cobordism from Λ to $\emptyset$, i.e. there is no exact Lagrangian "cap" of Λ.

Remark 12. Assume that Λ_- admits an exact Lagrangian filling L inside the symplectisation, and that ε^- is the augmentation induced by this filling. It follows that ε^+ is the augmentation induced by the filling $L \odot \Sigma$ of Λ_+ obtained as the concatenation of L and Σ. Using Seidel's isomorphisms

$$LCH^k_{\varepsilon^-,\varepsilon^-}(\Lambda^-) \cong H_{n-k}(L; R), \quad LCH^k_{\varepsilon^+,\varepsilon^+}(\Lambda^+) \cong H_{n-k}(L \odot \Sigma; R)$$

to replace the relevant terms in the long exact sequences (5.1) and (5.3), we obtain the long exact sequence for the pair $(L \odot \Sigma, L)$ and the Mayer-Vietoris long exact sequence for the decomposition $L \odot \Sigma = L \cup \Sigma$, respectively. This fact was already observed and used by the author in [69].

5.3 Restrictions on Lagrangian cobordisms

Using the long exact sequences from the previous subsection and their refinements to coefficients twisted by the fundamental group, we find strong topological restrictions on exact Lagrangian cobordisms between certain classes of Legendrian submanifolds.

5.3.1 The homology of an exact Lagrangian cobordism from a Legendrian submanifold to itself

From Theorem 11 we get the following result.

Theorem 13 ([24])**.** *Let* Σ *be an exact Lagrangian cobordism from* Λ *to* Λ *and* $\mathbb{F}$ *a field (of characteristic two if* Λ *is not relatively Pin). If the Chekanov-Eliashberg algebra* $\mathcal{A}(\Lambda; \mathbb{F})$ *admits an augmentation, then:*

(i) There is an equality $\dim_{\mathbb{F}} H_\bullet(\Sigma; \mathbb{F}) = \dim_{\mathbb{F}} H_\bullet(\Lambda; \mathbb{F})$*;*

(ii) The map $(i_*^-, i_*^+) : H_\bullet(\Lambda; \mathbb{F}) \to H_\bullet(\Sigma; \mathbb{F}) \oplus H_\bullet(\Sigma; \mathbb{F})$ *is injective; and*

(iii) The map $i_*^+ \oplus i_*^- : H_\bullet(\Lambda \sqcup \Lambda) \to H_\bullet(\Sigma)$ *is surjective.*

Here i^+ *is the inclusion of* Λ *as the positive end of* Σ*, while* i^- *is the inclusion of* Λ *as the negative end of* Σ*.*

A similar statement has been proven by Dimitroglou Rizell and the author in [37, Theorem 1.6] under the more restrictive assumption that Λ bounds an exact Lagrangian filling.

An immediate corollary of Theorem 9 is the following result, which had already appeared in [37, Theorem 1.7] under the stronger assumption that the negative end is fillable.

Theorem 14 ([24]). *If Λ is a homology sphere which admits an augmentation over $\mathbb{Z}$, then any exact Lagrangian cobordism Σ from Λ to itself is a homology cylinder (i.e. $H_\bullet(\Sigma, \Lambda) = 0$).*

Inspired by the work of Capovilla-Searle–Traynor [15], Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini and I prove the following restriction on the characteristic classes of an exact Lagrangian cobordism from a Legendrian submanifold to itself. Given a manifold M, we denote by $w_i(M)$ the i-th Stiefel-Whitney class of TM.

Theorem 15 ([24]). *Let Σ be an exact Lagrangian cobordism from Λ to itself, and $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Assume that $\mathcal{A}(\Lambda; \mathbb{F})$ admits an augmentation. If, for some $i \in \mathbb{N}$, $w_i(\Lambda) = 0$, then $w_i(\Sigma) = 0$.*

If Λ is relatively Pin, the same holds for the Pontryagin classes and for the Maslov class.

In particular, we provide a partially answer to Question 6.1 of the same article.

Corollary ([24]). If Λ is an orientable Legendrian submanifold admitting an augmentation, then any exact Lagrangian cobordism from Λ to itself is orientable.

5.3.2 Restrictions on the fundamental group of certain exact Lagrangian fillings and cobordisms

Following ideas of Sullivan in [100] and Damian in [33], Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini and I define a "twisted" version of the Cthulhu complex $Cth(\Sigma_0, \Sigma_1)$ with coefficients in the group ring $R[\pi_1(\Sigma_0)]$.

We also establish long exact sequences analogous to those in Section 5.2 involving homology groups over twisted coefficients in $R[\pi_1(\Sigma)]$. In the setting of Legendrian contact homology, these techniques were introduced by Ekholm and Smith in [55] and further developed by Eriksson-Östman in [60].

Using generalisations of the long exact sequence from Theorem 9 we prove the following theorem:

Theorem 16 ([24]). *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+. Assume that $\mathcal{A}(\Lambda^-; R)$ admits an augmentation and that Λ^+ has no Reeb chords in degree zero. If Λ^- and Λ^+ both are simply connected, then Σ is simply connected as well.*

We now present another result, whose proof uses an L^2-completion of the Floer homology groups with twisted coefficients and the L^2-Betti numbers of the universal cover (using results of Cheeger and Gromov in [25]).

Theorem 17 ([24]). *Let Λ be a simply connected Legendrian submanifold which is relatively Pin, and let Σ be an exact Lagrangian cobordism from Λ to itself. If $\mathcal{A}(\Lambda; \mathbb{C})$ admits an augmentation, then Σ is simply connected as well.*

Combining Theorem 14 with Theorem 17, Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini and I get the following result.

Corollary ([24]). Let Σ be an n-dimensional Legendrian homotopy sphere and assume that $\mathcal{A}(\Lambda; \mathbb{Z})$ admits an augmentation. Then any exact Lagrangian cobordism Σ from Λ to itself is an h-cobordism. In particular:

1. If $n \neq 3, 4$, then Σ is diffeomorphic to a cylinder;
2. If $n = 3$, then Σ is homeomorphic to a cylinder; and
3. If $n = 4$ and Λ is diffeomorphic to S^4, then Σ is diffeomorphic to a cylinder.

When $n = 1$, a stronger result is known. Namely, in [21, Section 4] we proved that any exact Lagrangian cobordism Σ from the standard Legendrian unknot Λ_0 to itself is compactly supported Hamiltonian isotopic to the trace of a Legendrian isotopy of Λ_0 which is induced by the complexification of a rotation by $k\pi$, $k \in \mathbb{Z}$. This classification makes use of the uniqueness of the exact Lagrangian filling of Λ_0 up to compactly supported Hamiltonian isotopy, which was proved in [59] by Eliashberg and Polterovich. In contrast, the methods we develop give restrictions only on the smooth type of the cobordisms and little information is known about their symplectic knottedness in higher dimension.

5.3.3 Obstructions to the existence of a Lagrangian concordance

First we note that a Lagrangian concordance is automatically exact. If Σ is a Lagrangian concordance, then $H_\bullet(\overline{\Sigma}, \partial_-\overline{\Sigma}; R) = 0$, and thus Theorem 9 implies the following corollary.

Corollary ([24]). Let Σ be an exact Lagrangian concordance from Λ^- to Λ^+. If, for $i = 0, 1$, ε_i^- is an augmentation of $\mathcal{A}(\Lambda^-; R)$ and ε_i^+ is the pull-back of ε_i^- under the DGA morphism induced by Σ, then the map

$$\Phi_\Sigma^{\varepsilon_0^-, \varepsilon_1^-} : LCH^\bullet_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda^-) \to LCH^\bullet_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+)$$

is an isomorphism. Consequently, there is an inclusion

$$\{LCH^\bullet_{\varepsilon_0^-, \varepsilon_1^-}(\Lambda^-)\}/\text{isom.} \hookrightarrow \{LCH^\bullet_{\varepsilon_0^+, \varepsilon_1^+}(\Lambda^+)\}/\text{isom.}$$

of the sets consisting of isomorphism classes of bilinearised Legendrian contact cohomologies, for all possible pairs of augmentations.

One can apply this corollary to obstruct the existence of exact Lagrangian concordances. For example, it can be applied to the computation of the linearised Legendrian contact homologies given by Chekanov in [26, Theorem 5.8] to prove that there is no exact Lagrangian concordance from either of the two Chekanov-Eliashberg knots to the other. We also use Corollary 5.3.3 to get new examples of non-symmetric concordances in the spirit of the example given by Chantraine in [20].

6. Front spinning construction and its applications

Front spinning is a procedure to construct a closed, orientable Legendrian submanifold $\Sigma_{S^1}\Lambda \subset \mathbb{R}^{2n+3}$ diffeomorphic to $\Lambda \times S^1$ from a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$. This procedure has been described by Ekholm, Etnyre and Sullivan in [46]. In [70] we define the notion of front S^m-spinning which produces a closed, orientable Legendrian submanifold $\Sigma_{S^m}\Lambda \subset \mathbb{R}^{2(n+m)+1}$ diffeomorphic to $\Lambda \times S^m$ from a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$. It is a natural generalization of the front S^1-spinning invented by Ekholm, Etnyre and Sullivan in [46]. We now describe the spherical spinning construction.

Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$ parameterized by $f_\Lambda : \Lambda \to \mathbb{R}^{2n+1}$ with

$$f_\Lambda(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$$

for $p \in \Lambda$. Without loss of generality assume that $x_1(p) > 0$ for all p.

We define $\Sigma_{S^m}\Lambda$ to be the Legendrian submanifold of $\mathbb{R}^{2(m+n)+1}$ parametrized by $f_{\Sigma_{S^m}\Lambda} : \Lambda \times S^m \to \mathbb{R}^{2(n+m)+1}$ with

$$\begin{aligned} f_{\Sigma_{S^m}\Lambda}(p,\theta,\overline{\phi}) = (\tilde{x}_{-m+1}(p,\theta,\overline{\phi}), \tilde{y}_{-m+1}(p,\theta,\overline{\phi}) \dots, &\tilde{x}_1(p,\theta,\overline{\phi}), \tilde{y}_1(p,\theta,\overline{\phi}), \\ &x_2(p), \dots, z(p)), \end{aligned}$$

where

$$\begin{cases} \tilde{x}_{-m+1}(p,\theta,\overline{\phi}) = x_1(p)\sin\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{x}_{-m+2}(p,\theta,\overline{\phi}) = x_1(p)\cos\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{x}_{-m+3}(p,\theta,\overline{\phi}) = x_1(p)\cos\phi_1 \dots \sin\phi_{m-1}, \\ \dots \\ \tilde{x}_1(p,\theta,\overline{\phi}) = x_1(p)\cos\phi_{m-1}, \end{cases} \tag{6.1}$$

$$\begin{cases} \tilde{y}_{-m+1}(p,\theta,\overline{\phi}) = y_1(p)\sin\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{y}_{-m+2}(p,\theta,\overline{\phi}) = y_1(p)\cos\theta \sin\phi_1 \dots \sin\phi_{m-1}, \\ \tilde{y}_{-m+3}(p,\theta,\overline{\phi}) = y_1(p)\cos\phi_1 \dots \sin\phi_{m-1}, \\ \dots \\ \tilde{y}_1(p,\theta,\overline{\phi}) = y_1(p)\cos\phi_{m-1}, \end{cases} \tag{6.2}$$

$\theta \in [0, 2\pi)$ and $\overline{\phi} = (\phi_1, \dots, \phi_{m-1}) \in [0,\pi]^{m-1}$.

Since Λ is a Legendrian submanifold of $\mathbb{R}^{2n+1}$ and hence $f_\Lambda^*(dz - \sum_{i=1}^n y_i dx_i) = 0$, we use Formulas 6.1 and 6.2 and see that

$$f^*_{\Sigma_{S^m}\Lambda}(dz - \sum_{i=-m+1}^{n} y_i dx_i) = 0.$$

Since $f_\Lambda(p) = (x_1(p), \dots, y_n(p), z(p))$, where $p \in \Lambda$, is a parametrization of an embedded n-dimensional submanifold and $x_1(p) > 0$ for all $p \in \Lambda$, we easily see that

$$f_{\Sigma_{S^m}\Lambda}(p,\theta,\overline{\phi}) = (\tilde{x}_{-m+1}(p,\theta,\overline{\phi}), \tilde{y}_{-m+1}(p,\theta,\overline{\phi}), \dots, \tilde{y}_1(p,\theta,\overline{\phi}), x_2(p), \dots, z(p)) \tag{6.3}$$

with $\theta \in [0, 2\pi)$, $\overline{\phi} = (\phi_1, \dots, \phi_{m-1}) \in [0, \pi]^{m-1}$ is a parametrization of an embedded $(n+m)$-dimensional submanifold. We prove the following extension of the spinning construction to exact Lagrangian cobordisms in [69] for $m = 1$ and in [70] for a general $m \geq 1$:

Theorem 18 ([69, 70]). *Let Λ^-, Λ^+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. If $\Lambda^- \prec_L^{ex} \Lambda^+$, then there exists an exact Lagrangian cobordism $\Sigma_{S^m} L$ such that $\Sigma_{S^m}\Lambda^- \prec_{\Sigma_{S^m} L}^{ex} \Sigma_{S^m}\Lambda^+$.*

The extension result in particular imply the following:

Theorem 19 ([70]). *There are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian $S^1 \times S^{i_1} \times \dots \times S^{i_k}$ in $\mathbb{R}^{2(\sum_{j=1}^k i_j + 1)+1}$ which have the same classical invariants if one of i_j's is odd.*

Dimitroglou Rizell and I prove that the front S^m-spinning construction preserves looseness.

Theorem 20 ([37]). *If Λ is a loose Legendrian submanifold of $\mathbb{R}^{2n+1}_{st}$, then $\Sigma_{S^m}\Lambda$ is a loose Legendrian submanifold of $\mathbb{R}^{2(n+m)+1}_{st}$.*

In [37], Dimitroglou Rizell and I prove certain forgetfullness properties of the front S^m-spinning construction. More precisely, we provide several types of examples of Legendrian submanifolds $\Lambda_1, \Lambda_2 \subset \mathbb{R}^{2n+1}_{st}$ which are not Legendrian isotopic, but which become Legendrian isotopic after we apply the front S^m-spinning construction.

Finally, in [38], Dimitroglou Rizell and I study the relation between the Chekanov-Eliashberg algebra of a Legendrian $\Lambda \subset \mathbb{R}^{2n+1}_{st}$ and the Chekanov-Eliashberg algebra of $\Sigma_{S^m}\Lambda$ and prove the following:

Theorem 21 ([38]). *The S^m-spun $\Sigma_{S^m}\Lambda \subset \mathbb{R}^{2(n+m)+1}_{st}$ of a Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}_{st}$ satisfies the following properties:*

(1) the Chekanov-Eliashberg algebra of $\Sigma_{S^m}\Lambda$ admits a (graded) augmentation to a unital commutative ring R if and only if that of Λ admits a (graded) augmentation to R;

(2) the characteristic algebra of $\Sigma_{S^m}\Lambda$ admits a (graded) finite-dimensional representation if and only if that of Λ admits a (graded) finite-dimensional representation;

(3) the Chekanov-Eliashberg algebra of $\Sigma_{S^m}\Lambda$ is acyclic if and only if that of Λ is acyclic.

6.1 Legendrian products and twist spuns

There are two constructions that are closely related to the front spinning construction: the twist spinning construction and the Legendrian product construction. First we recall the notion of Legendrian product.

Definition 22. Consider two Legendrian submanifolds

$$\iota_i \colon \Lambda_i \hookrightarrow (P_i \times \mathbb{R}, dz_i + \eta_i),\ \ i = 1, 2.$$

The *Legendrian product* $\Lambda_1 \boxtimes \Lambda_2 \looparrowright (P_1 \times P_2 \times \mathbb{R}, dz + \eta_1 + \eta_2)$ is the Legendrian immersion defined by

$$\iota_1 \boxtimes \iota_2(u_1, u_2) = (\Pi_{P_1}(\iota_1(u_1)), \Pi_{P_2}(\iota_1(u_1)), z_1(\iota_1(u_1)) + z_2(\iota_2(u_2)))$$

Legendrian products have been introduced by Lambert-Cole in [80], where computed their classical Legendrian invariants.

Then we recall the notion of twist spun.

Definition 23. Given a loop $\{\Lambda_\theta\}$, $\theta \in S^1$, of Legendrian embeddings of Λ in $(P \times \mathbb{R}, dz + \eta)$, the corresponding mapping torus has a natural Legendrian embedding

$$\Sigma_{S^1}\{\Lambda_\theta\} \subset (\mathbb{R}^2 \times P \times \mathbb{R}, dz - ydx + \eta)$$

called the *twist spun*

Twist spuns have first been constructed and studied by Ekholm–Kálmán in [51]. In the special case when the loop of Legendrians is constant, it recovers the S^1 front spinning.

With Dimitrolgou Rizell we construct examples of Legendrian products which are *not* Legendrian isotopic to twist spuns:

Theorem 24 ([40])**.** *There exist Legendrian product tori $\Lambda = \Lambda_1 \boxtimes \Lambda_2 \subset \mathbb{R}^5_{st}$ which are not Legendrian isotopic to any twist spun of a family of Legendrian knots.*

7. Arnold chord conjecture

7.1 Homological Arnold chord conjecture

One of the first striking applications of Gromov's theory of pseudoholomorphic curves [75] was that a closed exact Lagrangian immersion $\widetilde{\Lambda} \subset (P, d\theta)$ inside a Liouville manifold must have a double-point, given the assumption that it is Hamiltonian displaceable. Gromov's result has the following contact-geometric reformulation, which will turn out to be useful. The above result translates into the fact that a horizontally displaceable Legendrian submanifold Λ must have a *Reeb chord* for the above standard contact form. A similar result holds for Legendrian submanifolds of boundaries of subcritical Weinstein manifolds, as proven in [83] by Mohnke.

In the spirit of Arnold [7], the following conjectural refinement of the above result was later made: the number of Reeb chords on a chord-generic Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, dz + \theta)$ whose Lagrangian projection is Hamiltonian displaceable is at least $\frac{1}{2}\sum_i b_i(\Lambda; \mathbb{F})$. However, as was shown by Sauvaget in [94] by the explicit counter-examples inside the standard contact vector space $(\mathbb{R}^4 \times \mathbb{R}, dz + \theta_0)$, $\theta_0 = -(y_1 dx_1 + y_2 dx_2)$, the above inequality is not true without additional assumptions on the Legendrian submanifold; also, see the more recent examples constructed in [44] by Ekholm-Eliashberg-Murphy-Smith. The latter result is based upon the h-principle proven in [58] by Eliashberg-Murphy for Lagrangian cobordisms having loose negative ends in the sense of Murphy [84].

On the positive side, the above Arnold-type bound has been proven using the Legendrian contact homology of the Legendrian submanifold, under the additional assumption that the Legendrian contact homology algebra is sufficiently well-behaved. In the case when the Chekanov-Eliashberg algebra of a Legendrian admits an augmentation (this should be seen as a form of non-obstructedness for its Floer theory), the above Arnold-type bound was proven by Ekholm-Etnyre-Sullivan in [48] and by Ekholm-Etnyre-Sabloff in [45].

In [38], Georgios Dimitroglou Rizell and I generalised this proof to the case when the Chekanov-Eliashberg algebra admits a finite-dimensional matrix representation, in which case the same lower bound also is satisfied.

Theorem 25 ([38]). *Given a horizontally displaceable Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ with the property that its characteristic algebra $\mathcal{C}_\Lambda$ admits a k-dimensional representation $\rho : \mathcal{C}_\Lambda \to \mathrm{M}_k(\mathbb{F})$ for some field $\mathbb{F}$. Then the following inequality holds:*

$$\frac{1}{2}\sum_{i\in I} b_i \leq \sum_{i \in I} c_i,$$

where $b_j := \dim_{\mathbb{F}} H_j(\Lambda; \mathbb{F})$, c_j is a number of Reeb chords of Λ of grading j, and $I \in \{2\mathbb{Z}, \mathbb{Z} \setminus 2\mathbb{Z}\}$.

In addition, if ρ is a graded representation, then the following refinement of the previous inequality holds

$$b_i \leq c_i + c_{n-i},$$

for all n and i such that $n \geq i \geq 0$.

In addition, Dimitroglou Rizell and I in [38] constructed the following family of examples:

Theorem 26 ([38])**.** *There exists an infinite family of pairwise not Legendrian isotopic Legendrian submanifolds* $(\Lambda_l)_{l\in\mathbb{N}}$ *diffeomorphic to* $S^1 \times S^{m_1} \times \cdots \times S^{m_s}$ *such that the Chekanov-Eliashberg algebra* $\mathcal{A}(\Lambda_l)$ *does not admit an augmentation to any unital commutative ring, but the characteristic algebra* $\mathcal{C}_{\Lambda_l}$ *admits a finite-dimensional matrix representation over* $\mathbb{Z}_2$*, and hence*

$$\frac{1}{2}\sum_{i\in I} b_i \leq \sum_{i\in I} c_i(\Lambda_l),$$

where $b_j := \dim_{\mathbb{Z}_2} H_j(S^1 \times S^{m_1} \times \cdots \times S^{m_s}; \mathbb{Z}_2)$*,* $c_j(\Lambda_l)$ *is a number of Reeb chords of* Λ_l *of grading* j*, and* $I \in \{2\mathbb{Z}, \mathbb{Z} \setminus 2\mathbb{Z}\}$*.*

7.2 Failure of homological Arnold chord conjecture

There are examples of horizontally displaceable Legendrian submanifolds $\Lambda \subset P \times \mathbb{R}$ for which the homological Arnold's inequality is not satisfied. Observe that all known examples have an acyclic Chekanov-Eliashberg algebra.

Using this h-principle, together with the h-principle for exact Lagrangian caps as shown in [58], Ekholm, Eliashberg, Murphy and Smith in [44] provided many examples of exact Lagrangian immersions with few double-points. We here present a weaker form of their result.

Theorem 27 (Ekholm, Eliashberg, Murphy and Smith [44])**.** *Suppose that* Λ *is a smooth closed* n*-dimensional manifold for which* $T\Lambda \oplus \mathbb{C}$ *is a trivial complex bundle. There exists a loose horizontally displaceable Legendrian embedding* $\Lambda \subset \mathbb{R}^{2n+1}$ *satisfying*

$$\begin{cases} 1 \leq |\mathcal{Q}(\Lambda)| \leq 2, & \text{if } n \text{ is odd;} \\ |\mathcal{Q}(\Lambda)| = \frac{1}{2}|\chi(\Lambda)|, & \text{if } n \text{ is even and } \chi(\Lambda) < 0; \\ \frac{1}{2}|\chi(\Lambda)| \leq |\mathcal{Q}(\Lambda)| \leq \frac{1}{2}|\chi(\Lambda)| + 2, & \text{if } n \text{ is even and } \chi(\Lambda) > 0. \end{cases}$$

Dimitroglou Rizell and I show the following slight improvement of the lower bound in the case when a Legendrian submanifold has a non-acyclic Chekanov-Eliashberg algebra.

Theorem 28 ([38])**.** *Suppose that* $\Lambda \subset P \times \mathbb{R}$ *is a horizontally displaceable* n*-dimensional Legendrian submanifold whose characteristic algebra is non-trivial, but does not admit any finite-dimensional representations. It follows that*

$$|\mathcal{Q}(\Lambda)| \geq 3.$$

Moreover, if $n = 2k$*, we have the bound*

$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}|\chi(\Lambda)| + 2,$$

under the additional assumptions that Λ *is orientable and either*

1. $\chi(\Lambda) \geq 0$, *or*

2. $\mu(\Lambda) = 0$ *and all generators have non-negative grading.*

The above Arnold-type bound is also related to the one regarding the number of Hamiltonian chords between the zero-section in T^*L (or, more generally, any exact closed Lagrangian submanifold of a Liouville manifold) and its image under a generic Hamiltonian diffeomorphism. Namely, such Hamiltonian chords correspond to Reeb chords on a Legendrian lift of the union of the Lagrangian submanifold and its image under the Hamiltonian diffeomorphism. In fact, as shown by Laudenbach-Sikorav in [81], the number of such chords is bounded from below by the stable Morse number of the zero-section (and hence, in particular, it is bounded from below by half of the Betti numbers of the disjoint union of *two* copies of the zero-section). Arnold originally asked whether this bound can be improved, and if in fact the *Morse number* of the zero-section is a lower bound. However, this question seems to be out of reach of current technology. On the other hand, we note that the stable Morse number is equal to the Morse number in a number of cases, see [32].

In the following we assume that a Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, \alpha := dz + \theta)$ is chord-generic and has an exact Lagrangian filling $L_\Lambda \subset (\mathbb{R} \times P \times \mathbb{R}, d(e^t\alpha))$. Here t denotes the coordinate on the first $\mathbb{R}$-factor. Further, the set of Reeb chords c in degree $|c| = \mathrm{CZ}(c) - 1 \in \mathbb{Z}/\mathbb{Z}\mu_{L_\Lambda}$ will be denoted by $\mathcal{Q}_{|c|}(\Lambda)$, where the grading is induced by the Conley-Zehnder index modulo the Maslov number $\mu_{L_\Lambda} \in \mathbb{Z}$ of L_Λ as defined in [47]. Observe that $\mu_{L_\Lambda} = 0$ in particular implies that the first Chern class of $(P, d\theta)$ vanishes on $H_2(P)$.

For a group G being the epimorphic image of $\pi_1(L_\Lambda)$, consider the Morse homology complex $(CM_\bullet(L_\Lambda, f; R[G]), \partial_f)$ of L_Λ with coefficients in the group ring $R[G]$ twisted by the fundamental group, where R is a unital commutative ring and $f\colon L_\Lambda \to \mathbb{R}$ is a Morse function satisfying $df(\partial_t) > 0$ outside of a compact set. (The generators of this complex are graded by the Morse index, and the differential counts negative gradient flow lines.)

Theorem 29 ([39]). *Let $L_\Lambda \subset (\mathbb{R} \times P \times \mathbb{R}, d(e^t\alpha))$ be an exact Lagrangian filling of an n-dimensional closed Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, \alpha)$ with fundamental group $\pi := \pi_1(L_\Lambda)$ and Maslov number $\mu_{L_\Lambda} \in \mathbb{Z}$.*

(i) In the case when the filling is spin and when $\mu_{L_\Lambda} = 0$, the Morse homology complex $(CM_\bullet(L_\Lambda, f; \mathbb{Z}[\pi]), \partial_f)$ is simple homotopy equivalent to a $\mathbb{Z}[\pi]$-equivariant complex $(\mathbb{Z}[\pi]\langle \mathcal{Q}_{n-\bullet}(\Lambda)\rangle, \partial)$;

(ii) In the general case, it follows that the complex

$$(CM_\bullet(L_\Lambda, f; R[G]), \partial_f)$$

is homotopy equivalent in the category of G-equivariant complexes to a complex $(R[G]\langle \mathcal{Q}_{n-\bullet}(\Lambda)\rangle, \partial)$ with grading in $\mathbb{Z}/\mu_{L_\Lambda}\mathbb{Z}$. Here we can always take $R = \mathbb{Z}_2$, while we are free to choose an arbitrary unital commutative ring in the case when L_Λ is spin.

Now let $\operatorname{stabMorse}(M)$ denote the stable Morse number of a manifold M with possibly non-empty boundary. Using Theorem 29 and the adaptation of [32, Theorem 2.2] to the case of manifolds with boundary, the following result is immediate:

Corollary ([39]). Suppose that $\Lambda \subset P \times \mathbb{R}$ is a chord-generic closed Legendrian submanifold admitting an exact Lagrangian filling L_Λ which is spin and has vanishing Maslov number. It follows that the bound

$$|\mathcal{Q}(\Lambda)| \geq \text{stableMorse}(L_\Lambda) \tag{7.1}$$

is satisfied for the number of Reeb chords on Λ.

By using the long exact sequence in singular homology of the pair $(\overline{L_\Lambda}, \partial\overline{L_\Lambda} = \Lambda)$, where $\overline{L_\Lambda}$ denotes the compact part of L_Λ, we obtain the following inequalities

$$\text{stableMorse}(L_\Lambda) \geq \sum_i b_i(L_\Lambda; \mathbb{F}) \geq \frac{1}{2}\sum_i b_i(\Lambda; \mathbb{F}),$$

for any field $\mathbb{F}$. Obviously, Inequality (7.1) is a strengthening of the original Arnold-type bound.

In the course of showing the above result, Georgios Dimitroglou Rizell and I also obtain the following generalisation of the aforementioned result by Sikorav-Laudenbach [81], which also is related to the theory of stable intersection numbers as introduced by Eliashberg-Gromov in [57, Section 2.3].

Theorem 30. *[39] Consider a closed exact Lagrangian submanifold $L \subset (P, d\theta)$ which is spin and has vanishing Maslov number. For any $k \geq 0$, the exact Lagrangian submanifold $L \times \mathbb{R}^k \subset (P \times \mathbb{C}^k, d\theta \oplus \omega_0)$ with $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_k \wedge dy_k$ satisfies the property that*

$$\#(L \times \mathbb{R}^k) \pitchfork \phi^1_{H_s}(L \times \mathbb{R}^k) \geq \text{stableMorse}(L),$$

given that the above intersection is transverse, and that the Hamiltonian is of the form $H_s = f_s + Q$, where:

- *$Q(x_1+iy_1, \ldots, x_k+iy_k) = Q(x_1, \ldots, x_k)$ is a non-degenerate quadratic form on $\mathbb{R}^k \subset \mathbb{C}^k$; and*
- *$f_s \colon P \times \mathbb{C}^k \to \mathbb{R}$, $s \in [0,1]$, satisfies that $\max_{s\in[0,1]} \|f_s\|_{C^1}$ is bounded for a product Riemannian metric of the form $g_P \oplus g_{\text{std}}$ on $P \times \mathbb{C}^k$. Here we moreover require g_P to be invariant under the Liouville flow on $(P, d\theta)$ outside of a compact subset, while g_{std} denotes the Euclidean metric.*

We also get the following two theorems which are consequences of Theorem 29 together with the algebraic machinery developed by Ono and Pajitnov in [90]. For a finitely presented group G, we denote by $d(G) \in \mathbb{Z}_{\geq 0}$ the minimal number of generators of G and in [39] prove the following:

Theorem 31 ([39]). *Let $\mu_{L_\Lambda} = 0$. Assume that $\pi_1(L_\Lambda)$ admits a finite epimorphic image G, which is a simple or solvable group.*

(i) Under the above assumptions, we have

$$|\mathcal{Q}(\Lambda)| \geq d(G) + \sum_{i \neq 1} b_i(L_\Lambda; \mathbb{F});$$

(ii) If moreover $\pi_1(L_\Lambda)$ is a finite perfect group, then

$$|\mathcal{Q}(\Lambda)| \geq d(G) + \sum_{i \neq 1} b_i(L_\Lambda; \mathbb{F}) + 2.$$

Here we have to use the field $\mathbb{F} = \mathbb{Z}_2$ unless L_Λ is spin, in which case it can be chosen arbitrarily.

Theorem 32 ([39]). *Assume that $\pi_1(L_\Lambda)$ admits a finite epimorphic image G, which is a simple or solvable group.*

(i) Under the above assumptions, we have

$$|\mathcal{Q}(\Lambda)| \geq \max(1, d(G) - 1) + \sum_{i \neq 1} b_i(L_\Lambda; \mathbb{F}),$$

where $i \in \mathbb{Z}/\mu_{L_\Lambda}\mathbb{Z}$;

(ii) If moreover $\mu_{L_\Lambda} \geq 2n + 2$, then

$$|\mathcal{Q}(\Lambda)| \geq d(G) + \sum_{i \neq 1} b_i(L_\Lambda; \mathbb{F}),$$

where $i \in \mathbb{Z}/\mu_{L_\Lambda}\mathbb{Z}$.

Here we have to use the field $\mathbb{F} = \mathbb{Z}_2$ unless L_Λ is spin, in which case it can be chosen arbitrarily.

8. Polyfilable Legendrian submanifolds

Within the last few years the question of existence of infinitely many exact Lagrangian fillings for Legendrian links has received a certain amount of attention. First it has been positively answered by Casals and Gao [16], and then by An-Bae-Lee [4, 5], Casals-Zaslow [18], and Gao-Shen-Weng [67, 68].

In [17] Casals and Ng following the ideas of Kálmán [76] have constructed the first series of Legendrian links in the standard contact 3-dimensional vector space with the property that the Legendrian contact homology DGAs detect infinitely many exact Lagrangian fillings. We show that S^k fron spinning construction applied to the examples of Casals–Ng leads to examples of Legendrian submanifolds with infinite number of pairwise Hamiltonian non-isotopic exact Lagrangian fillings. More precisely, we prove the following:

Theorem 33 ([72]). *For a given $m \geq 1$ and $k_i \geq 2$, where $i = 1, \dots, m$, there is a Legendrian submanifold Λ in the standard contact vector space $\mathbb{R}^{2(k_1+\cdots+k_m+1)+1}$ diffeomorphic to the disjoint union of some number of $S^1 \times S^{k_1} \times \cdots \times S^{k_m}$ which admits an infinite number of exact Lagrangian fillings distinct up to Hamiltonian isotopy.*

Then from the examples in [72] and polyfillable Legendrian submanifolds from the work of Cao-Gallup-Hayden-Sabloff [14] we constructed the following family:

Theorem 34 ([71]). *For a given $n \geq 3$ and $K > 0$, there is a connected Legendrian submanifold $\Lambda \subset (\mathbb{R}^{2n+1}, dz - \sum_i y_i dx_i)$ which admits a collection of exact Lagrangian fillings*

$$\{L_k^j \mid 1 \leq j \leq K(\Lambda), k \in \mathbb{Z}_{\geq 0}\},$$

where $K(\Lambda) \geq K$, such that

- *$L_{k_1}^{j_1}$ is not homeomorphic to $L_{k_2}^{j_2}$ for $j_1 \neq j_2$ and $k_1, k_2 \in \mathbb{Z}_{\geq 0}$;*
- *for a fixed j, $\{L_k^j\}_{k=1}^{\infty}$ consists of an infinite number of diffeomorphic exact Lagrangian fillings that are pairwise distinct up to Hamiltonian isotopy.*

9. Torsion in Legendrian contact homology

Legendrian contact homology of a Legendrian submanifolds $\Lambda \subset \mathbb{R}^{2n+1}$ is often defined with $\mathbb{Z}_2$-coefficients, but if Λ is spin, it can be also defined over other fields, over $\mathbb{Z}$ [48, 78] and even more general coefficient rings such as $\mathbb{Z}_2[H_1(\Lambda;\mathbb{Z})]$ or $\mathbb{Z}[H_1(\Lambda;\mathbb{Z})]$ [47, 78].

The Legendrian contact homology DGA is not finite rank, even in fixed degree; the same holds in homology. In order to deal with this issue Chekanov in [26] proposed to use an augmentation of the DGA to produce a generically finite-dimensional linear complex, whose homology is called linearized Legendrian contact homology.

Given an exact Lagrangian filling L of Λ in the symplectization of $\mathbb{R}^{2n+1}$ with vanishing Maslov number, it induces the augmentation of the Chekanov–Eliashberg algebra, i.e. a unital DGA homomorphism $\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}_2, 0)$, see [52]. If besides that L is equipped with a spin structure extending the given spin structure on Λ, then one also has an augmentation $\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}, 0)$, see [52, 78].

The majority of computations of linearized Legendrian contact homology groups have been done with $\mathbb{Z}_2$-coefficients. One can ask whether for coefficients in $\mathbb{Z}$ one can get a non-trivial algebraic torsion in linearized Legendrian contact cohomology. Note that certain examples of Legendrian submanifolds of $J^1(T^n)$ and of $J^1(S^2)$ with torsion elements in linearization appear in [48, 86, 87]. We answer the more concrete question about the type of algebraic torsion that could appear for Legendrian submanifolds of $\mathbb{R}^{2n+1}$, i.e. whether an arbitrary finitely generated abelian group can be realized as a linearized Legendrian contact cohomology of some Legendrian submanifold of the standard contact vector space.

In [73], we provide the following answer to this question in high dimensions:

Theorem 35 ([73]). *Given a finitely generated abelian group G and an integer $n \geq 3$, $n \neq 4$. There is a Legendrian submanifold Λ in $\mathbb{R}^{2n+1}$ of Maslov number 0 such that the Chekanov-Eliashberg algebra of Λ admits an augmentation $\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{Z}, 0)$ with $LCH^{n-1}_{\varepsilon}(\Lambda;\mathbb{Z}) \simeq G$.*

Very recently this question has been also solved by Lipshitz and Ng in the low dimensional case [82], i.e. for Legendrian submanifolds of $\mathbb{R}^3$.

10. Non-geometric augmentations in high dimensions

There are a few obstructions to the existence of an exact Lagrangian filling which induces a given augmentation, see [19, 43, 36, 53, 61, 66, 69]. Besides that there are many examples of augmentations of Legendrian knots that are not induced by exact Lagrangian fillings i.e., are *non-geometric*.

We construct non-geometric augmentations for certain high dimensional Legendrian submanifolds. The obstructions to the existence of exact Lagrangian fillings that we use are Seidel's isomorphism [43, 36] and the injectivity of a certain algebraic map between the corresponding augmentation varieties proven by Gao and Rutherford [66]. More precisely, in [74] we prove the following:

Theorem 36. *[74] There is a Legendrian submanifold* Λ *in* $\mathbb{R}^{2n+1}_{st}$ *of Maslov number* 0 *such that the Chekanov-Eliashberg algebra of* Λ *admits an augmentation* $\varepsilon : \mathcal{A}(\Lambda) \to (\mathbb{F}_2, 0)$ *which is not induced by a spin exact Lagrangian filling of Maslov number* 0*.*

In addition, along the way in [74] we prove the following relation between augmentation varieties over a field $\mathbb{F}$ of Legendrian submanifolds and their spherical spuns:

Theorem 37. *[74] Let* Λ *be a Maslov number* 0 *spin Legendrian submanifold, whose Reeb chords have only non-negative degrees. Then there is an isomorphism of (graded) augmentation varieties* $Aug(\Lambda; \mathbb{F}) \simeq Aug(\Sigma_{S^m}\Lambda; \mathbb{F})$ *for all* $m \geq 2$*, and* $Aug(\Sigma_{S^1}\Lambda; \mathbb{F}) \simeq Aug(\Lambda; \mathbb{F}) \times \mathbb{F}^*$*.*

11. Noncommutative augmentation category

In order to recover at least part of the nonlinear information lost in the linearisation of Legendrian contact homology, one can study products in the linearised Legendrian contact homology groups induced by the product structure of the Chekanov-Eliashberg DGA.

Civan, Koprowski, Etnyre, Sabloff and Walker in [29] endowed Chekanov's linearised chain complex with an A_∞-structure. This construction was generalised in [13] by Chantraine and Bourgeois, who showed that a differential graded algebra naturally produces an A_∞-category whose objects are its augmentations. In dimension three, the A_∞-category constructed by Chantraine and Bourgeois admits a unital refinement defined by Ng, Rutherford, Shende, Sivek and Zaslow in [89]. The latter article also establishes an equivalence between this unital A_∞-category and one defined in terms of derived sheaves of microlocal rank one with microsupport given by a fixed Legendrian knot.

A_∞-algebras are by now classical structures which were first introduced by Stasheff in [98] as a tool in the study of 'group-like' topological spaces. Fukaya was the first to upgrade the notion of an A_∞-algebra to that of an A_∞-category. In [64] he associated an A_∞-category, which now goes under the name of the *Fukaya category*, to a symplectic manifold. See [95] for a good introduction. Inspired by Fukaya's work [64], Kontsevich in [79] formulated the *homological mirror symmetry conjecture* relating the derived Fukaya category of a symplectic manifold to the derived category of coherent sheaves on a "mirror" manifold.

The construction in [29] and [13] defines A_∞-operations only when the coefficient ring of the DGA is commutative. In [22] Chantraine, Dimitroglou Rizell, Ghiggini and I extend that construction to noncommutative coefficient rings in the following two cases:

(I) the coefficients of the DGA as well as the augmentations are taken in a unital noncommutative algebra, or

(II) the coefficients of the DGA as well as the augmentations are taken in a noncommutative *Hermitian algebra*. This case includes both finite-dimensional algebras over a field and group rings.

Case (II) is obviously included in Case (I), but as we show in [22] there is a particularly nice alternative construction of an A_∞-structure in case (II) which gives a different result. Both generalisations above are sensible to study when having Legendrian isotopy invariants in mind, albeit for different reasons.

Case (I) occurs because there are Legendrian submanifolds whose Chekanov-Eliashberg DGA does not admit augmentations in any unital algebra of finite rank over a commutative ring, but admits an augmentation in a unital noncommutative infinite-dimensional one (for example, in their characteristic algebras). The first such examples were Legendrian knots constructed by Sivek in [96] building on examples found by Shonkwiler and Shea Vela-Vick in [97]. From them, Dimitroglou Rizell and I constructed higher dimensional examples in [38]. Observe that any differential graded algebra has an augmentation in its "characteristic algebra", introduced by Ng in [85], which is the quotient of the DGA by the two-sided ideal generated by its boundaries. This algebra is in general noncommutative and infinite-dimensional, and any augmentation factors through it. It is of course possible that the characteristic algebra vanishes, but it does so if and only if the DGA is acyclic [35]. The *complex* that we will define in case (I) (but not the higher order operations) was used in [34] by Dimitroglou Rizell in order to deduce that a Legendrian submanifold with a non-acyclic Chekanov-Eliashberg DGA does not admit a displaceable Lagrangian cap.

Finally, we note that the construction we give in Case (I) is closely related to the A_∞-structures and bounding cochains with noncommutative coefficients as introduced by Cho, Hong and Sui-Cheong in their work [27]. Namely, the (uncurved) A_∞-structures that we produce from a DGA and its augmentations can be seen to coincide with the (uncurved) A_∞-structures produced by their bounding cochains.

Case (II) also occurs naturally in the context of Legendrian contact homology. For example, in [88] Ng and Rutherford show that augmentations of certain satellites of Legendrian knots induce augmentations in matrix algebras for the Chekanov-Eliashberg DGA of the underlying knot. Moreover, coefficients in a group ring appear naturally if one considers the Chekanov-Eliashberg DGA with coefficients "twisted" by the fundamental group of the Legendrian submanifold. We learned this construction from Eriksson-Östman, who makes use of it in [60]. This version of Legendrian contact homology can be seen as a natural generalisation of Morse homology and Floer homology with coefficients twisted by the fundamental group; see the work [100] by Sullivan and [33] by Damian. In the setting of Legendrian contact homology with twisted coefficients, an exact Lagrangian filling gives rise to an augmentation taking values in the group ring of the fundamental group of the filling. See the work [24] by Chantraine, Dimitroglou Rizell, Ghiggini and I for more details, were Legendrian contact homology with twisted coefficients is used to study the topology of Lagrangian fillings and cobordisms.

In [22] we outline how our construction can be used as an efficient computational tool for distinguishing a Legendrian knot from its Legendrian mirror in the case when there are no augmentations in a commutative algebra. Note that it, in general, it is much easier to extract invariants from the A_∞-algebra compared to the DGA.

Finally, we recall that Legendrian contact homology is not the only place where noncommutative graded algebras appear in symplectic geometry. Another source is cluster homology, a proposed generalisation of Lagrangian Floer homology due to Cornea and Lalonde [30], which is supposed to provide an alternative approach to the A_∞-structures in Floer homology introduced by Fukaya, Oh, Ohta and Ono [65].

12. The wrapped Fukaya category of Weinstein manifolds and sectors and Seidel's conjecture

A Fukaya category of a symplectic manifold (M, ω) is a category $\mathcal{F}(M)$ whose objects are Lagrangian submanifolds of M and morphisms are Floer chain groups: $Hom(L_0, L_1) = CF(L_0, L_1)$. Its finer structure can be described in the language of quasi categories as an A_∞-category. As Fukaya categories are A_∞-categories, they have associated derived categories, which are the subject of the celebrated homological mirror symmetry conjecture of Maxim Kontsevich [79] which says that for a closed symplectic Calabi-Yau manifold, its Fukaya category is derived equivalent to the category of coherent sheaves on its mirror. Recall that a phenomenon called mirror symmetry was first observed by physicists studying string theory, and it provides a bridge between symplectic geometry and string theory.

The version of Fukaya category for Weinstein manifolds (manifolds with cylindrical ends) was defined by Abouzaid and Seidel [2] and it is called the wrapped Fukaya category. There is a version of homological mirror symmetry for Weinstein manifolds, where wrapped Fukaya categories play the same role as Fukaya categories in the homological mirror symmetry conjecture in the case of closed manifold. The following result that I prove with Baptiste Chantraine, Georgios Dimitroglou Rizell and Paolo Ghiggini provides a natural presentation of the wrapped Fukaya category of a Weinstesin manifold (or, even more generally, Weinstein sector):

Theorem 38 ([23]). *The wrapped Fukaya category of a Weinstein manifold (or, more generally, Weinstein sector) W is generated by the Lagrangian cocore discs of critical Weinstein handles.*

In other words, in a joint work with Chantraine, Dimitroglou Rizell and Ghiggini [23], we decompose any object in the wrapped Fukaya category as a twisted complex built from the cocores of the critical (i.e. half-dimensional) handles in a Weinstein handle decomposition. The main tools used are the Floer homology theories of exact Lagrangian immersions, of exact Lagrangian cobordisms in the SFT sense (i.e. between Legendrians), as well as relations between these theories. Note that exact Lagrangians admit Legendrian lifts, and that appropriate Lagrange surgeries can be seen to give rise to an exact Lagrangian cobordism of the aforementioned type.

As a corollary of Theorem 38 we get a proof of the long-standing Seidel's conjecture:

Theorem 39. *[23] Let W be a Weinstein manifold of finite type. Let $\mathcal{D}$ be the full A_∞-subcategory of the wrapped Fukaya category $\mathcal{W}F(W)$ whose objects are the Lagrangian cocore planes. Then the open-closed map*

$$\mathcal{O}C \colon HH_*(\mathcal{D}, \mathcal{D}) \to SH^*(W).$$

is an isomorphism. Hence,

$$\mathcal{OC}\colon HH_*(\mathcal{W}F(W), \mathcal{W}F(W)) \to SH^*(W)$$

is an isomorphism. Here HH_* *denotes Hochschild homology,* SH^* *denotes symplectic cohomology and* $\mathcal{OC}$ *is the open-closed map defined in [1].*

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