Halfspace depth: Theory and computation

Habilitation Thesis

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Contents

Introduction: Halfspace depth and multivariate quantiles

Ranks, order statistics, and quantiles are at the core of nonparametric statistics. For a Borel probability measure P on $\mathbb{R}$, the cumulative distribution function (c.d.f.)

$$F\colon \mathbb{R} \to [0,1]\colon x \mapsto P\left((-\infty, x]\right) \tag{1}$$

is well known to represent the measure P uniquely. Its (generalized) inverse function

$$Q\colon [0,1] \to \mathbb{R}\colon \delta \mapsto \inf\left\{x \in \mathbb{R}\colon F(x) \geq \delta\right\} \tag{2}$$

is called the *quantile function* of P. Quantiles are certainly among the key concepts of statistics. As the simplest example, take the median of P in $\mathbb{R}$, defined as the $1/2$-quantile $Q(1/2)$ of P. The median has an array of fine properties: (i) it is always well defined (unlike the expectation, for example), (ii) it is straightforward to interpret, and (iii) is quite stable, in the sense that perturbations of P do not much disturb its median. It is also straightforward to observe that just like the c.d.f. F, also the quantile function Q describes the measure P uniquely. The measure P is simple to recover directly from its quantile function Q, or its c.d.f. F. All these traits make the quantiles a tool that suits statistics and data analysis exceptionally well. The abundant field of nonparametric statistics attests to this.

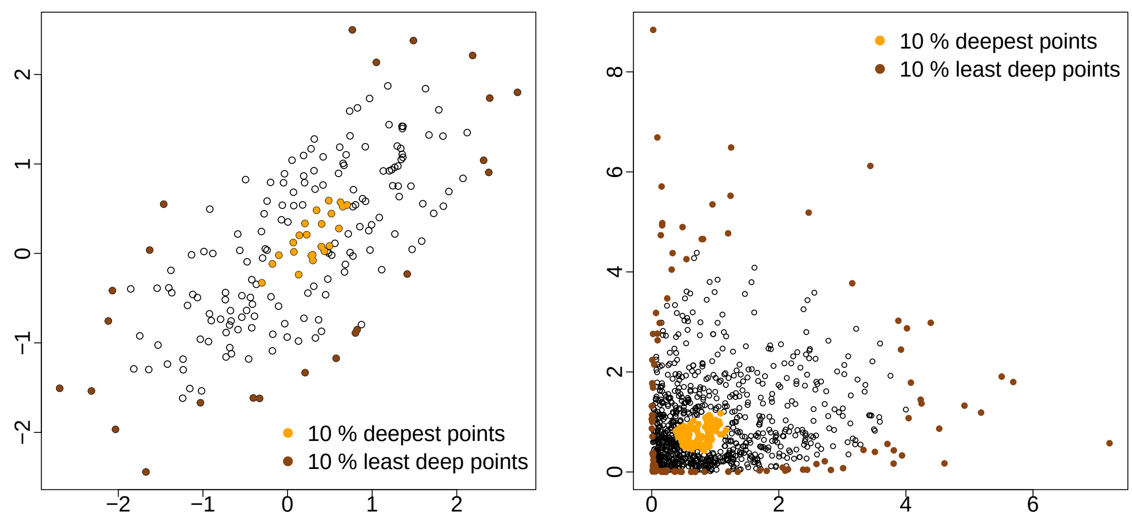

Figure 1: Two datasets in $\mathbb{R}^2$ — each point represents a single datum. For the empirical measure given by each of these datasets, the halfspace depth D of all sample points was computed. $10\ \%$ of observations with the highest and lowest depth values are plotted in yellow and brown, respectively, in both panels. The depth induces a data-dependent ordering of $\mathbb{R}^d$ in the center-outwards sense — points of high depth form a "center" of the dataset; low depth points occupy the data cloud's outskirts.

Therefore, the situation with probability measures living in $\mathbb{R}$ is quite simple. For the analysis of $\mathbb{R}$-valued data, one may work with the measure P on $\mathbb{R}$ directly, or alternatively with its representation via the c.d.f. F, or the quantile function Q. All this is possible due to the natural ordering of the real line $\mathbb{R}$, implicitly used in both (1) and (2).

The situation with data living in multidimensional spaces $\mathbb{R}^d$, or spaces $\mathcal{X}$ that may lack linear structure (non-Euclidean spaces), is far more complicated. This thesis focuses on the linear spaces $\mathbb{R}^d$ with $d > 1$. No unique canonical notion of total ordering exists in $\mathbb{R}^d$, and while the c.d.f. of a Borel probability measure P in $\mathbb{R}^d$ is typically defined as

$$F \colon \mathbb{R}^d \to [0,1] \colon \ (x_1, \dots, x_d) \mapsto P\left((-\infty, x_1] \times \cdots \times (-\infty, x_d]\right),$$

this function does not possess a natural inverse that would play the role of the quantile function. For this reason, quantiles, and by extension, nonparametric statistical techniques, cannot be used in multivariate spaces in a straightforward way.

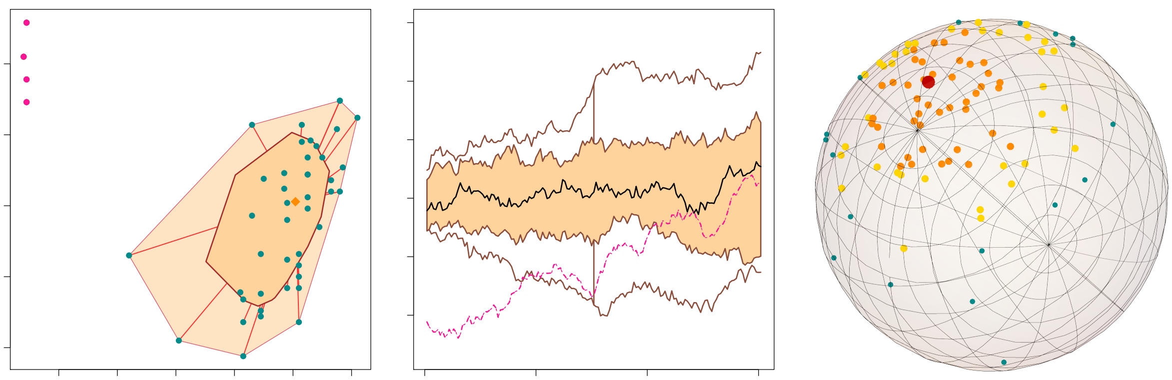

Figure 2: (a) The bagplot – a bivariate version of the boxplot from $\mathbb{R}$ — of a dataset in $\mathbb{R}^2$ [83], (b) a functional boxplot [99], and (c) a heatplot for spherical data [97]. All these graphical methods rely on depth, and are part of standard R packages [81] used by thousands of users. The theory behind them is, however, far from trivial.

Our intention is to study *statistical depth functions* (usually called just *depths*, for simplicity). Depths have been proposed to serve as generalizations of quantiles, ranks, and orderings for data living in multivariate spaces $\mathbb{R}^d$, or spaces $\mathcal{X}$ without linear structure. Formally speaking, denote $\mathcal{P}(\mathcal{X})$ the set of all Borel probability measures on a topological space $\mathcal{X}$. Depth is a function

$$D \colon \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to [0, \infty) \colon \ (x; P) \mapsto D(x; P)$$

that to any point $x \in \mathcal{X}$ and a measure $P \in \mathcal{P}(\mathcal{X})$ assigns $D(x; P)$, the depth of x with respect to (w.r.t.) P. Taking P as a fixed given measure, the depth of x quantifies how much "centrally positioned" the point x is within the geometry induced by the probability mass of P. The higher the depth of x is, the more central x is, while $D(x; P)$ close to zero signifies that x is located on the periphery of P, or is an observation that is atypical for P. Any depth function $D(\cdot; P) \colon \mathcal{X} \to [0, \infty) \colon x \mapsto D(x; P)$ induces the (depth-based) *central regions* of P, defined as the upper level sets of $D(\cdot; P)$

$$D_\delta(P) = \{x \in \mathcal{X} \colon D(x; P) \geq \delta\}, \qquad \text{for } \delta \geq 0. \tag{3}$$

These central regions form a system of nested sets, non-increasing with growing $\delta \geq 0$, and $D_0(P) = \mathcal{X}$. The depth D induces a P-dependent ordering of the points from $\mathcal{X}$; $x \in \mathcal{X}$ is deeper than $y \in \mathcal{X}$ if and only if $D(x;P) > D(y;P)$, see Figure 1. The central regions (3) are used to visualize this ordering and provide a base for working with the depth of P explicitly. Depths have found many applications in nonparametric statistics of more complex data; for several applications in different setups see Figure 2.

The depth function is supposed to generalize the quantile function (2) to $\mathcal{X}$-valued data. The role of the median $Q(1/2)$ from $\mathbb{R}$ is played by the deepest point $x \in \mathcal{X}$, defined as (any) point $x \in \mathcal{X}$ that maximizes the depth function $D(\cdot;P)$. Equivalently, the depth-induced median is any point x inside the smallest non-empty central region (3). The latter set is sometimes called the *depth median set* of P. The other central regions (3) act as sets of points that are sufficiently centrally positioned. They are analogs of inter-quantile regions

$$[Q(1/2-\beta), Q(1/2+\beta)] \subset \mathbb{R} \tag{4}$$

with suitable $\beta \in [0,1/2]$ for Q the quantile function (2) of $P \in \mathcal{P}(\mathbb{R})$. For $\beta = 1/4$, for instance, we obtain the smallest region of the most centrally positioned points of P-mass at least $1/2$. This is the *box*, commonly used in the construction of the boxplot for $\mathbb{R}$-valued data. Its depth-based analog for $P \in \mathcal{P}(\mathcal{X})$ based on the central regions (3) is called the *bag* and can be used for the construction of a version of the boxplot for $\mathcal{X}$-valued data, see Figure 2 (a, b).

There is no universally agreed-upon notion of depth — instead, depth comes in many flavors. Many depth functions were proposed in the literature. For excellent accounts on the general theory of statistical depth functions, we refer to [53, 106, 62, 59, 60, 63, 65].

In this thesis, we focus on the classical *halfspace depth* (sometimes called also *Tukey depth*, or *location depth*) proposed by J. W. Tukey [100]. It is defined in $\mathcal{X} = \mathbb{R}^d$ with $d \geq 1$. The *halfspace depth* of $x \in \mathbb{R}^d$ w.r.t. $P \in \mathcal{P}(\mathbb{R}^d)$ is

$$D(x;P) = \inf_{u \in \mathbb{S}^{d-1}} P(H_{x,u}), \tag{5}$$

where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \colon \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^d$, and $H_{x,u} = \{y \in \mathbb{R}^d \colon \langle x-y, u\rangle \leq 0\}$ is the closed halfspace whose boundary hyperplane passes through $x \in \mathbb{R}^d$ with inner unit normal $u \in \mathbb{S}^{d-1}$. In words, the halfspace depth of a point x is the minimum P-mass of a halfspace that contains x.

Tukey proposed the halfspace depth in 1975. Its first rigorous study, however, comes only with D. L. Donoho [16, 17] in the 1980s, who coined the term *depth*. Soon after, H. Oja proposed what is now known as the *simplicial volume depth* [78], and R. Y. Liu brought up the notion of the *simplicial depth* in $\mathbb{R}^d$ [50, 51] and $\mathcal{X} = \mathbb{S}^{d-1}$ [52]. This triggered an immense surge of interest in statistical depth functions and their applications to data analysis. In the coming decades, dozens of depth functions have been proposed for $\mathbb{R}^d$-valued measures, and in more general spaces $\mathcal{X}$. The halfspace depth (5), however, remains a prime example of a statistical depth function. Still, almost 50 years after its introduction, it rouses the attention of statisticians, and many open problems continue to stimulate interest in this depth and the associated statistical procedures.

The halfspace median was recently observed to play a crucial role in robust statistics of multivariate data: It is the minimax optimal estimator of location in common robustness models [7], by means of which it lately generated a great deal of attention in both statistics [13, 39, 55] and machine learning [14, 15]. Likewise, other central regions $D_\delta(P)$ of the halfspace depth have been recently well studied as multivariate analogs of the quantiles of P [5, 29, 35, 42, 82, 106].

The overarching theme of this thesis is the mathematical study of the halfspace depth in $\mathbb{R}^d$. It is based on the following seven papers:

(A) Nagy, S., Schütt, C., and Werner, E. M. (2019). Halfspace depth and floating body. *Stat. Surv.*, 13:52–118, [76].

(B) Nagy, S. (2020). The halfspace depth characterization problem. In *Nonparametric statistics*, volume 339 of Springer Proc. Math. Stat., pages 379–389. Springer, Cham, [69].

(C) Nagy, S. (2021). Halfspace depth does not characterize probability distributions. *Statist. Papers*, 62(3):1135–1139, [71].

(D) Laketa, P. and Nagy, S. (2021). Reconstruction of atomic measures from their halfspace depth. *J. Multivariate Anal.*, 183:Paper No. 104727, 13 pages, [46].

(E) Nagy, S. and Dvořák, J. (2021). Illumination depth. *J. Comput. Graph. Statist.*, 30(1):78–90, [73].

(F) Nagy, S., Dyckerhoff, R., and Mozharovskyi, P. (2020). Uniform convergence rates for the approximated halfspace and projection depth. *Electron. J. Stat.*, 14(2):3939–3975, [75].

(G) Dyckerhoff, R., Mozharovskyi, P., and Nagy, S. (2021). Approximate computation of projection depths. *Comput. Statist. Data Anal.*, 157:Paper No. 107166, 23 pages, [24].

All these papers deal with the mathematics of the halfspace depth. They can be split thematically into two major parts. The first four papers (A)–(D) are mostly theoretical. We point to many surprising connections of the halfspace depth with concepts from pure mathematics, and leverage these interdisciplinary links to resolve several long-standing open problems of the theory of halfspace depth. Perhaps the most interesting is the main result of paper (C), which resolves the 30-year-old *halfspace depth characterization conjecture* in the negative. Paper (E) brings a practical application of the unveiled links between depth and convex geometry; we use the geometric concept of illumination to improve halfspace depth in data analysis.

The second part of the present work, described in papers (F) and (G), tackles the practical problem of efficient computation of the halfspace depth. We are given a point $x \in \mathbb{R}^d$, and a collection of data points $X_1, \dots, X_n \in \mathbb{R}^d$ that are typically realizations of a random sample from $P \in \mathcal{P}\left(\mathbb{R}^d\right)$. How to compute the halfspace depth of x w.r.t. the measure that assigns equal mass $1/n$ to each of the points X_i, $i = 1, \dots, n$? While the general problem of computing the halfspace depth exactly is known to be NP-hard [40], a substantial body of research has been focused on the task of finding algorithms for both exact and approximate computation of the halfspace depth. In (F) and (G), we scrutinize some of the approximation algorithms and propose new techniques for the computation of the halfspace depth. Our goal is to provide guidelines on how to compute the halfspace depth in practice.

Perhaps the most exciting feature of the present work is its interdisciplinarity. The methods we use to study the halfspace depth seldom originate only in statistics; we extensively engage mathematical methods previously unexplored in multivariate statistics to solve statistical problems. Novel meanings and interpretations are attached to ideas, especially from geometry, and original applications toward the practice of data analysis are established. At this point, we already know that our observed connections of the halfspace depth with concepts of more general mathematics are not an exception. We managed to couple several other notions frequently employed in multivariate statistics with ideas from geometry, analysis, or machine learning. We refer to papers (i) **[68, 70]**[1] for the geometry of the scatter halfspace depth defined in [79], (ii) **[104]** for an analysis of the h-depth from [11, 12] in view of the recent advances in machine learning, and (iii) **[72]** for glimpses into the mathematics of the simplicial depth [50, 51]. In the present thesis, we focus exclusively on the halfspace depth and offer an array of novel insights into its behavior and properties.

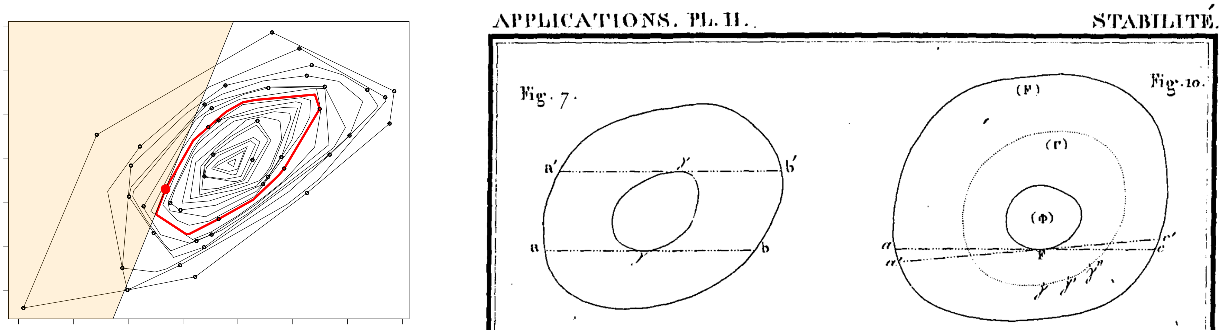

Figure 3: Left: A dataset P_n in $\mathbb{R}^2$ of 45 points (black) with the contours of $D_\delta(P_n)$ (black polygons), and $x \in \mathbb{R}^2$ (red point). We have $D(x; P_n) = 8/45$; a halfspace $H_{x,u}$ that attains this depth is the lightly colored region. The central region $D_{8/45}(P_n)$ is the polygon with a red boundary. Right: An illustration from Dupin (1822) [19]. The regions inside the convex bodies K in this figure (e.g., the region (Φ) on the right hand side) are equivalent with $D_\delta(P)$, for P uniform on K.

Notations and terminology

As basic references, we take [18] for probability, [92] for statistics, and [87] for geometry. $\mathcal{P}\left(\mathbb{R}^d\right)$ are all Borel probability measures on $\mathbb{R}^d$, and $\mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d \colon \|x\| = 1\right\}$ is the unit sphere. For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, we write $X \sim P$ if X is a random vector distributed as P; $\stackrel{d}{=}$ means "is equal in distribution". We call a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ *empirical* if there exists an integer $n \geq 1$ and a collection of (not necessarily distinct) points $x_1, \dots, x_n \in \mathbb{R}^d$ such that P can be written as a sum of n Dirac masses at the points $x_1, \dots, x_n$, each with total mass $1/n$. A measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is called *finitely atomic* if the support of P consists of a finite point set; a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is *atomic* if its support is at most countable. Every finitely atomic measure is atomic, and every empirical measure is finitely atomic; a finitely atomic measure is empirical if and only if all P-masses of the atoms of P are rational numbers.

A halfspace is a set $H_{x,u} = \left\{y \in \mathbb{R}^d \colon \langle x - y, u\rangle \leq 0\right\}$ for $x \in \mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$; the collection of all halfspaces in $\mathbb{R}^d$ is $\mathcal{H}$. The set of convex bodies, i.e., compact convex subsets $K \subset \mathbb{R}^d$ with non-empty interior, is denoted by $\mathcal{K}^d$. In a convenient departure from the common terminology, we say that $H \in \mathcal{H}$ *supports* K if the interior of H does not intersect K, but the boundary of H does. We say that K has a *smooth boundary* ∂K at $x \in \partial K$ if there exists a unique halfspace $H \in \mathcal{H}$ that supports K and contains x; the convex body K is *smooth* if its boundary is smooth at every point.

[1] References to papers of the author other than (A)–(G) are in **bold**.

1 Summary of paper (A): Halfspace depth and floating body

In the extensive semi-survey (A), we approach the halfspace depth from two different angles: (i) we introduce the maximum halfspace depth $\max_{x \in \mathbb{R}^d} D(x; P)$ as a measure of symmetry of the measure $P \in \mathcal{P}(\mathbb{R}^d)$, and (ii) draw an equivalence between the central regions $D_\delta(P)$ and the floating bodies. The history of both measures of symmetry and floating bodies in convex geometry is much longer than the history of depth in statistics. While the halfspace depth was introduced in the 1970s [100], measures of symmetry have been studied since the 1910s [2], and floating bodies can be traced well into the early 19th-century [19], see Figure 3. Interestingly, the related advances from geometry have never been adequately recognized or leveraged in the statistics literature. Our intention in paper (A) was to bridge this gap between fields of mathematics and explore the common ground of all these seemingly little related theories.

Maximum depth as a measure of symmetry

In the first major part of paper (A) in Section 4, we couple the halfspace depth and the associated median with the so-called *measures of symmetry* of convex bodies, a concept that has been covered extensively in convex geometry [34]. Standardly, a measure of symmetry σ is a map that to a convex body $K \in \mathcal{K}^d$ assigns a quantity $\sigma(K) \in [0, 1]$ that assesses the degree of symmetry of K; $\sigma(K) = 1$ if and only if K is symmetric (i.e., $K = -K$). It turns out that, at least when the uniform distribution $P \in \mathcal{P}(\mathbb{R}^d)$ on a convex body $K \in \mathcal{K}^d$ is identified with K itself, the maximum halfspace depth $\max_{x \in \mathbb{R}^d} D(x; P)$ exactly coincides with a measure of symmetry of K called the *Winternitz measure of symmetry*. The history of the Winternitz measure of symmetry goes back to the 1910s [2]. In that period, Arthur Winternitz proved, speaking in the terminology of the halfspace depth, that $\max_{x \in \mathbb{R}^2} D(x; P) \geq 4/9$ for any $P \in \mathcal{P}(\mathbb{R}^2)$ uniform on $K \in \mathcal{K}^2$, and that this lower bound is attained if and only if K is a triangle. In Section 4 of (A), we gather many little-recognized connections between the research on depth in statistics and the related results on measures of symmetry. We draw from the latter and refine bounds on the maximum halfspace depth w.r.t. measures $P \in \mathcal{P}(\mathbb{R}^d)$ with specific structural properties. Interestingly, we uncovered novel results not only by applying advances from geometry toward statistics. In our treatment of symmetries of measures, we found that in multivariate statistics, remarkable results directly applicable to geometry can be found. A case in point is the Funk theorem on symmetric convex bodies, which we introduce next.

Symmetries of measures: The Funk theorem and beyond.

In 1915 Paul Funk [30] proved the following result:

Theorem. *For a convex body $K \in \mathcal{K}^3$, if every plane passing through the origin splits K into two parts of equal volume, then K must be symmetric around the origin (i.e., $K = -K$).*

In $\mathbb{R}^2$ an analogous result is easy to see; Funk conjectured that it is true for all dimensions $d \geq 1$. The complete resolution of Funk's conjecture remained elusive for 50 years. Only in the 1970s, an involved proof based on spherical harmonics emerged in geometry [86]. Since that time, somewhat simpler proofs of the Funk theorem have appeared in the literature [33], but no truly elementary proof for convex bodies has been known in geometry.

Not observing those difficulties, systematic research of symmetric measures in $\mathbb{R}^d$ was initiated in statistics in the late 1990s [107] by Zuo and Serfling. Symmetries of measures are more delicate than those of convex bodies. Following [107], we say that a $X \sim P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is

1. *halfspace symmetric* if there exists $x \in \mathbb{R}^d$ such that $D(x; P) \geq 1/2$;
2. *angularly symmetric* if for some $x \in \mathbb{R}^d$ we have[2] $(X - x)/\|X - x\| \overset{d}{=} -(X - x)/\|X - x\|$;
3. *centrally symmetric* if $(X - x) \overset{d}{=} -(X - x)$.
4. *spherically symmetric* if $X \overset{d}{=} O\,X$ for any orthogonal matrix $O \in \mathbb{R}^{d \times d}$.

It is known that spherical ⇒ central ⇒ angular ⇒ halfspace symmetry, with no implication possible to reverse. In [85] we find

Theorem. *For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ non-atomic the halfspace symmetry and the angular symmetry coincide.*

The proof in [85] is surprisingly elementary; it uses only the Cramér-Wold device, a tool known since the 1930s [8]. As a particular case, for P uniform on a convex body $K \in \mathcal{K}^d$, this theorem covers Funk's conjecture and appears to be its first easily accessible proof in the literature. Curiously, neither Zuo and Serfling [107] nor Rousseeuw and Struyf [85] were aware of Funk's result when proving this claim in the language of depth.

An early predecessor of halfspace depth: Dupin's floating body

In the second main part of paper (A), we provide another unlikely pairing between the multivariate statistics of depth and convexity theory. We show that the concept of floating bodies, studied in geometry since the early 19th century, is entirely equivalent to the central regions $D_\delta(P)$ of measures P. This will allow us to call $D_\delta(P)$ the floating body of P. The abundance of results known about floating bodies has never been considered in light of their statistical applications.

It is well known [57, 82, 106] that the central regions (3) of the halfspace depth can be written as

$$D_\delta(P) = \bigcap \left\{ H \in \mathcal{H} \colon P\left(H\right) > 1 - \delta \right\} \quad \text{for } \delta \geq 0. \tag{6}$$

The equality in (6) follows directly from (5); it is not difficult to see that the halfspace depth may be alternatively defined only by means of its level sets in (6) for all $\delta \geq 0$ [22].

The equivalent expression for the central regions $D_\delta(P)$ in (6) has already been well established in convex geometry in the early 19th century. In 1822, French geometer Charles Dupin [19] published a treatise on the mechanics of solid bodies floating in fluids. One of the concepts he considered, the *floating body* of $K \in \mathcal{K}^d$, is defined as a convex subset $K_{[\delta]}$ of K such that each supporting hyperplane of $K_{[\delta]}$ cuts off a set of volume δ from K, see Fig. 4 (a). Suppose that K is a solid of unit volume and volumetric mass density $\delta \in (0, 1/2)$. This body is placed into water — a halfspace with normal

[2] Here we define $0/0 = 0$ with the origin $0 \in \mathbb{R}^d$ on the right hand side in case there is non-zero P-mass at x.

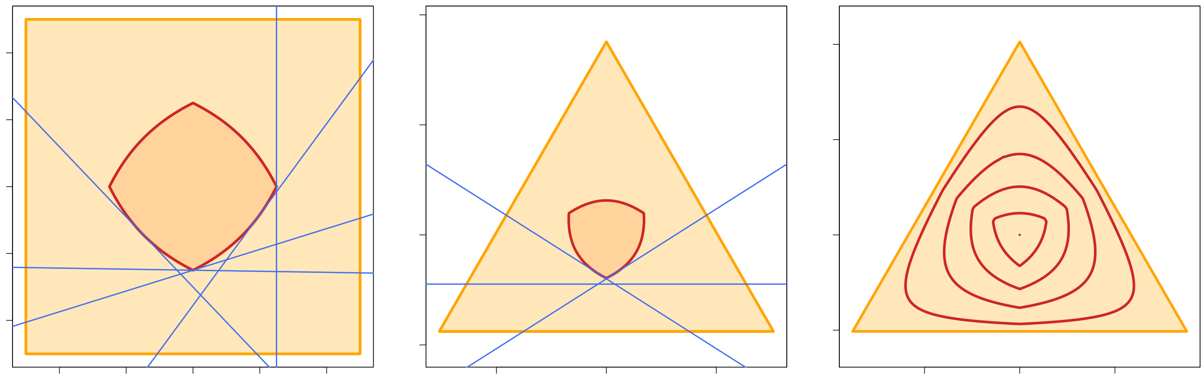

Figure 4: (a) A (Dupin's) floating body (inner colored body) of the uniform distribution P on a square in $\mathbb{R}^2$ with $\delta = 0.25$, with several halfplanes of P-mass δ (boundaries in blue). (b) The uniform measure P on a triangle in $\mathbb{R}^2$ does not possess Dupin's floating bodies. The central region $D_\delta(P)$ for $\delta = 0.3$ (inner colored body), with several halfplanes with P-mass δ (boundaries in blue). One of the halfplanes does not support $D_\delta(P)$. (c) Several contours of the halfspace depth for P uniform on a triangle. Each contour has three non-smooth points.

$u \in \mathbb{S}^{d-1}$. If the water is of unit density, the Archimedean principle tells that for K to float, part of K of volume δ must be submerged. The floating body $K_{[\delta]}$ is the part of K that remains above the water surface in every direction. The body $K_{[\delta]}$ equals the upper level set (6) of the depth for P uniform on K, provided that the former exists. In particular, floating bodies carry information about the depth of P, and vice versa. Floating bodies gained importance in geometry, mainly thanks to their connections with the crucial *affine surface area* [2]. Naturally, floating bodies have thus been thoroughly investigated in geometry; we draw from that research and apply it to multivariate statistics.

The *Dupin floating body of a measure* $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ at level $\delta > 0$ may be defined as a convex set $D_{[\delta]}(P)$ such that each supporting halfspace $H \in \mathcal{H}$ of $D_{[\delta]}(P)$ carries mass $P(H) = \delta$. It is not difficult to observe that if $D_{[\delta]}(P)$ exists, then $D_{[\delta]}(P) = D_\delta(P)$. This is proved formally in Proposition 31 of paper (A).

The greatest difficulty with the Dupin floating body is that it does not have to exist. The simplest example is the uniform distribution $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ on a triangle, see Fig. 4 (b), where the Dupin floating body does not exist for any level $\delta > 0$. The problem with the possible non-existence of Dupin's floating bodies in geometry was resolved in 1990 by Schütt and Werner [89], who introduced the *convex floating body* that, to $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, assigns the expression in (6) directly. In complete isolation from the research in statistics, the depth (5) in this way also resurfaced in geometry [1, 89]. The connections of the depth to the more regular Dupin's floating bodies, or the affine invariants derived from them, have been unknown in statistics until very recently. Progress came with our collaboration with Carsten Schütt and Elisabeth M. Werner, prominent geometers and the original authors of the modern version of the floating body. Together we covered this gap in mathematics in paper (A). In what follows we pick several interesting novel observations that can be found in our treatment of floating bodies and the depth in paper (A).

Regular depth asymptotics/Existence and smoothness of Dupin's floating bodies

The following problem is crucial for both theory and applications of the halfspace depth:

Problem. *For an empirical measure P_n of a random sample $X_1, \dots, X_n$ generated from $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, consider the depth process $\zeta(x) = \sqrt{n}\left(D(x; P_n) - D(x; P)\right)$. When is ζ asymptotically Gaussian?*

By [56], the depth process $\zeta(x)$ is asymptotically Gaussian if and only if the infimum in (5) is attained by a single halfspace $H \in \mathcal{H}$. The following geometric criterion can be shown to be equivalent, as observed, e.g., in **[32]**.

Lemma. *The depth process $\zeta(x)$ is asymptotically Gaussian if and only if $D_\delta(P)$ with $\delta = D(x; P)$ has a smooth boundary at x.*

The problem of finding P with smooth boundaries of $D_\delta(P)$ is hard; see also Fig. 4. Very little progress has been made in that direction in statistics in the past 30 years. In fact, several authors have explicitly argued that no distributions other than elliptically symmetric distributions[3] are known to possess central regions with smooth boundaries [57, 56, 43, 42, 5]. We were surprised to find out that this problem is well studied in convex geometry. The following theorem can be found in [58], where it is stated in terms of Dupin's floating bodies of convex bodies.

Theorem. *Let $K \in \mathcal{K}^d$ be a symmetric convex body in $\mathbb{R}^d$ and let $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ be the uniform distribution on K. Then all Dupin's floating bodies $D_{[\delta]}(P)$ with $\delta \in (0, 1/2)$ exist. If, in addition, K is smooth and strictly convex, then the boundary of each $D_{[\delta]}(P)$, $\delta \in (0, 1/2)$, is smooth.*

This elegant result brings a whole new class of distributions with smooth central regions — uniform distributions on symmetric smooth strictly convex bodies possess smooth central regions. Thus, their depth processes are asymptotically normal everywhere except at the center of symmetry. This fundamental result has apparently not been known in statistics before.

Uniform distributions on convex bodies are, however, not the only measures whose Dupin's floating bodies are known to exist. Analogous results can be found in the literature on s-concave measures. Recall that a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is said to be *s-concave* for $-\infty \leq s \leq 1$, if

$$P\left(\lambda A + (1-\lambda)B\right) \geq \begin{cases} \min\left\{P(A), P(B)\right\} & \text{for } s = -\infty, \\ P(A)^\lambda P(B)^{1-\lambda} & \text{for } s = 0, \\ \left(\lambda P(A)^s + (1-\lambda)P(B)^s\right)^{1/s} & \text{otherwise,} \end{cases}$$

for all non-empty Borel sets $A, B \subseteq \mathbb{R}^d$ and all $\lambda \in [0,1]$. We say that P is *strictly s-concave* if the inequality above is always strict whenever $A \neq B$. A measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is known to be s-concave with $s \leq 1/d$ if and only if P has a density f that is supported on an open convex subset U of $\mathbb{R}^d$ and that is $s_d = s/(1 - d\,s)$-concave, i.e., for all $x, y \in U$ and for all $\lambda \in [0,1]$,

$$f\left(\lambda x + (1-\lambda)y\right) \geq \begin{cases} \min\left\{f(x), f(y)\right\} & \text{for } s = -\infty, \\ f(x)^\lambda f(y)^{1-\lambda} & \text{for } s = 0, \\ \left(\lambda f(x)^{s_d} + (1-\lambda)f(y)^{s_d}\right)^{\frac{1}{s_d}} & \text{otherwise.} \end{cases}$$

[3] Or, slightly more generally, affine images of α-symmetric distributions, see the discussion on our paper (C) below.

For $s = 0$, s-concave measures are also called log-concave measures, and represent a natural generalization of uniform measures on convex bodies; indeed, any uniform measure on a convex body is log-concave. As proved in [3], any (centrally) symmetric s-concave measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ with $s \geq -1$ does possess all Dupin's floating bodies $D_{[\delta]}(P)$ for $\delta \in (0, 1/2)$. This observation is stated as Theorem 30 in paper (A), with a broad discussion on its statistical implications.

While symmetric s-concave measures do possess Dupin's floating bodies, it appears to be an open problem whether (strictly) symmetric s-concave distributions with smooth densities possess smooth floating bodies. An even more pressing open problem concerns the existence, and the smoothness of, Dupin's floating bodies of P without the assumption of symmetry of P. It is non-trivial to find examples of asymmetric measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ whose (single) central region $D_\delta(P)$ is Dupin's, or smooth **[70]**.

Characterization and reconstruction: A positive result

In Section 8 of paper (A), we first delve into the crucial *halfspace depth characterization conjecture*, a long-standing open problem on the behavior of the depth (5). The halfspace depth is designed with the intention of generalizing quantiles to $\mathbb{R}^d$-valued data. In $\mathbb{R}$, we argued that the quantiles uniquely characterize elements of $\mathcal{P}(\mathbb{R})$, and to a quantile function Q from (2), it is simple to find its unique measure $P \in \mathcal{P}(\mathbb{R})$. Thus, it is immaterial whether P itself or its quantiles are used for the analysis. The same questions for the depth (5) are known as the *characterization problems* [98, 43]:

Characterization: Does for each $P \neq P'$ in $\mathcal{P}\left(\mathbb{R}^d\right)$ exist $x \in \mathbb{R}^d$ such that $D(x; P) \neq D(x; P')$?

Reconstruction: How to find $P(B)$ for $B \subset \mathbb{R}^d$ Borel from the function $x \mapsto D(x; P)$ only?

Characterization problems are essential to the theory of depth; the halfspace depth is a universally valid representative of measures in $\mathbb{R}^d$ only if the characterization conjecture is true. The characterization/reconstruction problems have been open for decades, with several partial positive results available. For example, it is known that empirical measures in $\mathbb{R}^d$ are uniquely characterized by their halfspace depth $D(\cdot; P)$ [98]; we detail several additional relevant results from the literature in our summary of papers (B), (C), and (D) below.

In Theorem 34 in (A), we use Dupin's floating bodies of measures to formulate the most general characterization result for the halfspace depth that can be found in the literature. Our result is based on the assumption that all Dupin's floating bodies of $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ exist, that is

$$\text{the Dupin floating body } D_{[\delta]}(P) \text{ of } P \in \mathcal{P}\left(\mathbb{R}^d\right) \text{ exists for all } \delta \in (0, 1/2). \tag{7}$$

It turns out that (7) is equivalent to the possibility of determining the P-mass of each closed halfspace $H \in \mathcal{H}$ directly from its halfspace depth in a particularly simple way. To state our claim, first observe that since the Dupin floating body is assumed to exist for δ arbitrarily close to $1/2$, the measure P must be halfspace symmetric. Indeed, if the depth δ of a halfspace median of P is smaller than $1/2$, we take $\gamma \in (\delta, 1/2)$. Our assumption (7) then yields a contradiction because necessarily $D_\gamma(P) = \emptyset$, meaning that also $D_{[\gamma]}(P) = \emptyset$, and no closed halfspace with P-mass γ can support $D_{[\gamma]}(P)$. Knowing that any measure P satisfying (7) is halfspace symmetric, we can use [107, Theorem 2.1] to claim that the center of the halfspace symmetry of P must be a unique point $x_P \in \mathbb{R}^d$, unless we are in the trivial case when P is supported in a line in $\mathbb{R}^d$ with more than one (univariate) median. Returning to our main result, in Theorem 34 in paper (A), it is proved that (7) is equivalent with the two conditions (i) P is smooth in the sense of attaching $P(\partial H) = 0$ to the boundary hyperplane ∂H of each halfspace $H \in \mathcal{H}$, and (ii) for each halfspace $H \in \mathcal{H}$ that contains x_P we can write

$$P(H) = 1 - \sup_{x \in \partial H} D(x; P). \tag{8}$$

Taking into account that P must be smooth, formula (8) allows us to reconstruct the measure P from its halfspace depth completely. Indeed, if a halfspace $H \in \mathcal{H}$ contains the point of maximum depth $x_P \in \mathbb{R}^d$, we know $P(H)$ directly from (8); if $x_P \notin H$, we just set $P(H) = \sup_{x \in \partial H} D(x; P)$. In particular, we are able to determine the P-mass of each closed halfspace in $\mathbb{R}^d$, and using the classical Cramér-Wold device [8], the measure P is determined uniquely among all measures that satisfy (7).

Our reconstruction procedure generalizes the main result from [43]. That result claims that if all regions $D_\delta(P)$ have smooth boundaries, then P can be uniquely determined from its halfspace depth. Indeed, by Proposition 31 in paper (A), we know that a central region $D_\delta(P)$ with a smooth boundary is a Dupin's floating body of P, and (7) applies. In particular, we obtain that any elliptically symmetric distribution is uniquely characterized by its halfspace depth function, with an explicit reconstruction formula (8) available. Several other types of distributions that satisfy (7) will be given in the summary of paper (C) below.

2 Summary of paper (B): The halfspace depth characterization problem

Paper (B) offers a broad analysis of the available partial positive results toward the halfspace depth characterization conjecture. It discusses eight positive results that can be found in the literature. The following papers claim that a measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is uniquely characterized by its halfspace depth function $x \mapsto D(x; P)$ if

1. **Struyf and Rousseeuw (1999):** P is empirical [98];
2. **Koshevoy (2002):** P is finitely atomic [44];
3. **Hassairi and Regaieg (2007):** P is finitely atomic [36], using a simpler proof than [44];
4. **Cuesta-Albertos and Nieto-Reyes (2008):** P is atomic [10];
5. **Koshevoy (2003):** P satisfies a certain moment condition (in particular, if P has bounded support) [45];
6. **Hassairi and Regaieg (2008):** P has a sufficiently smooth density [37];
7. **Kong and Zuo (2010):** P has all central regions $D_\delta(P)$ smooth [43]; and finally,
8. **Nagy, Schütt and Werner (2019):** P has all Dupin's floating bodies (A).

Putting all these results together, it may appear that the general halfspace depth characterization conjecture is valid in most interesting situations. It has been explicitly conjectured that any Borel probability measure is characterized by its halfspace depth [10, 42]. In our paper (B) we, however, argue that the situation is much more complicated. We proceed case-by-case and comment on each of the results above.

Finitely atomic measures

The original characterization result for empirical measures [98] and its extension to the finitely atomic case [44] are both truly pioneering advances. They will be commented on in detail in the description of our paper (D) refining those findings. The simplified proof of Koshevoy's characterization theorem [44] from [36] is, however, incomplete.[4] The proof of Theorem 5 from [36] intends to show that if two finitely atomic measures P and P' from $\mathcal{P}\left(\mathbb{R}^d\right)$ have the same halfspace depth, then $P = P'$. It proceeds by induction, recovering all the atoms of P and P', starting from the atom x_1 of the smallest depth $D(\cdot;P)$. This is performed in formulas (3.1) and (3.2) in [36]. In the induction step, it is, however, assumed that also after the common atom x_1 of P and P' is removed from the support of P and P', the halfspace depths of the restrictions of P and P' to the set $\mathbb{R}^d \setminus \{x_1\}$ are identical. This does not directly follow from the assumption of the theorem, as removing x_1 from the support of P and P' will change the depths of the remaining points in $\mathbb{R}^d$. For a sound argument, one has to use the slightly more complicated argument from [44] or its simplified version explained in the summary of paper (D) below.

Atomic and boundedly supported measures

Two of the strongest characterization results from [10] and [45] have both incomplete proofs, and these proofs do not appear to be possible to be fixed. We explain these difficulties in detail in Section 3 of (B). This leaves the halfspace depth characterization conjecture open for general atomic measures, and in the interesting case of distributions with bounded support.

Measures with smooth density

The condition of the main result from [37] is usually misinterpreted in the literature. In the proof of the main Theorem 3.2 from [37] it is assumed that (i) the density f of P is continuous in the interior of its support, and (ii) an intractable condition (H) involving certain derivatives of a function related to f holds. When interpreting this result in the literature, it is however claimed that Theorem 3.2 from [37] requires only (i) "absolute continuity of P" [101], (ii) "absolute continuity with connected support" [20], or (iii) "absolute continuity with connected support and continuous density function in the interior of the support" [96]. In fact, also in the original paper [37], it is stated that condition (H) holds true

"when the probability density function of the distribution has partial derivatives in the interior of its support." (9)

[4] This is not disclosed in paper (B), as it came to our knowledge only after paper (B) was published.

In (B) we argue that the last statement is false, and the condition (H) may also be violated for distributions with infinitely smooth densities. Interestingly, the technical condition (H) can be rephrased, for general distributions P without the need to impose absolute continuity. It can be stated as the geometric requirement that every supporting halfspace H of every central region $D_\delta(P)$ carries probability mass exactly $P(H) = \delta$. In other words, extending the characterization result from [37], we arrive precisely at the requirement of the existence of Dupin's floating bodies, which is our result described in the summary of paper (A). For the additional claim (9) from [37], we offer a direct rebuttal that will be described in the summary of our paper (C) below.

Smooth central regions and existing Dupin's floating bodies

The final positive results on the halfspace depth characterization conjecture from [43] claim that if all central regions $D_\delta(P)$ have smooth boundaries, then P can be recovered from its halfspace depth function. As argued in the summary of our paper (A), this condition is stronger than our requirement of the existence of Dupin's floating bodies of P. Nevertheless, there are not many types of measures that satisfy these conditions. The only examples known appear to be (centrally) symmetric s-concave measures with $s \geq -1$, and α-symmetric measures with $\alpha > 1$.

Summary of positive results

Our previous discussion reduces the set of conditions under which $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is known to be characterized by its halfspace depth to:

- measures in $\mathbb{R}$, where the c.d.f. of $P \in \mathcal{P}(\mathbb{R})$ can be recovered from its halfspace depth directly;
- finitely atomic measures in $\mathcal{P}\left(\mathbb{R}^d\right)$ [98, 44]; and
- measures P in $\mathcal{P}\left(\mathbb{R}^d\right)$ whose all Dupin's floating bodies $D_{[\delta]}(P)$ exist with $\delta \in (0, 1/2)$.

In particular, it currently seems to be unknown whether measures with infinitely many atoms, or even the uniform distribution on a triangle in $\mathbb{R}^2$, are characterized by their halfspace depth!

3 Summary of paper (C): Halfspace depth does not characterize probability distributions

In paper (C) we resolve the general characterization conjecture for the halfspace depth in the negative. We construct uncountable families of different probability measures in $\mathcal{P}\left(\mathbb{R}^d\right)$ for each integer $d > 1$ with identical halfspace depth. In our construction, we used α-symmetric measures [26, 28, 41].

For $\alpha > 0$, denote the $\mathcal{L}_\alpha$-norm[5] of $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ by $\|t\|_\alpha^\alpha = \sum_{i=1}^d |t_i|^\alpha$ for $\alpha \in (0, \infty)$, and $\|t\|_\infty = \max_{i=1,\dots,d} |t_i|$. Following [26, 28] we say that $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is α-*symmetric* if its characteristic function

$$\psi(t) = \int_{\mathbb{R}^d} \exp\left(i \langle x, t \rangle\right) \mathrm{d}\, P(x)$$

[5] Also for $\alpha \in (0, 1)$ we refer to this function as a norm for brevity.

takes the form $\psi(t) = \xi(\|t\|_\alpha)$ for some function $\xi\colon [0,\infty) \to \mathbb{R}$. Let $X = (X_1, \dots, X_d) \sim P$ be a random vector distributed as P. Directly from the expression for the characteristic function of X, we see that

$$P \text{ is } \alpha\text{-symmetric} \iff \langle X, u\rangle \overset{d}{=} \|u\|_\alpha X_1 \text{ for all } u \in \mathbb{R}^d. \tag{10}$$

The case $\alpha = 2$ covers the spherically symmetric distributions introduced in our summary of paper (A), see also [93]. The α-symmetric measures are fascinating, yet somewhat mysterious objects. Still, in 1990 [28, Section 7.4], it was unknown whether such a measure even exists in dimension $d > 2$, with $\alpha > 2$.

Starting from (10), it is easy to derive that for any α-symmetric (probability) measure $X \sim P$ we can determine the halfspace depth exactly. Indeed, if $(\Omega, \mathcal{F}, \mathsf{P})$ is the probability space on which X is defined, we can write

$$\begin{aligned} D(x; P) &= \inf_{u \in \mathbb{S}^{d-1}} \mathsf{P}(\langle X, u\rangle \leq \langle x, u\rangle) = \inf_{u \in \mathbb{S}^{d-1}} \mathsf{P}(\|u\|_\alpha X_1 \leq \langle x, u\rangle) \\ &= \mathsf{P}\left(X_1 \leq \inf_{u \in \mathbb{S}^{d-1}} \langle x, u\rangle / \|u\|_\alpha\right) = F_1\left(-\|x\|_\beta\right) \end{aligned} \tag{11}$$

for β the conjugate index to α, and F_1 the c.d.f. of X_1. The last equality in (11) is essentially the generalized Hölder inequality

$$-\|u\|_\alpha \|x\|_\beta \leq \langle x, u\rangle,$$

which holds true for any $\alpha > 0$, if the conjugate index β is defined by

$$1/\alpha + 1/\beta = 1 \text{ if } \alpha > 1, \text{ and } \beta = \infty \text{ if } \alpha \leq 1. \tag{12}$$

Of course, the Hölder inequality is sharp in the sense that equality is attained for each $x \in \mathbb{R}^d$ by taking a specific choice of the direction $u \in \mathbb{S}^{d-1}$.

We consider the case $\alpha \leq 1$. That leads to $X \sim P$ with extremely heavy-tailed distributions, for which only the first marginal X_1 of X determines the depth

$$D(x; P) = F_1(-\|x\|_\infty) \quad \text{for all } x \in \mathbb{R}^d.$$

When rewritten in this way, the characterization problem for the halfspace depth translates to the problem of finding α-symmetric measures P with $\alpha \leq 1$ with a single identical marginal. Such measures are known to exist since the 1930s [88]. Fix $\gamma \in (0,1)$ and take

$$\psi_\alpha(t) = \exp(-\|t\|_\alpha^\gamma) \tag{13}$$

for $\gamma \leq \alpha \leq 1$. Then

- Measure $P_\alpha \in \mathcal{P}(\mathbb{R}^d)$ with characteristic function ψ_α exists [88];
- The conjugate index to $\alpha \leq 1$ is $\beta = \infty$; and
- For the characteristic function of X_1 with $X \sim P_\alpha$ we have

  $$\mathrm{E}\exp(\mathrm{i}\, t\, X_1) = \exp(-|t|^\gamma) \quad \text{for all } t \in \mathbb{R},$$

  i.e. F_1 does not depend on α.

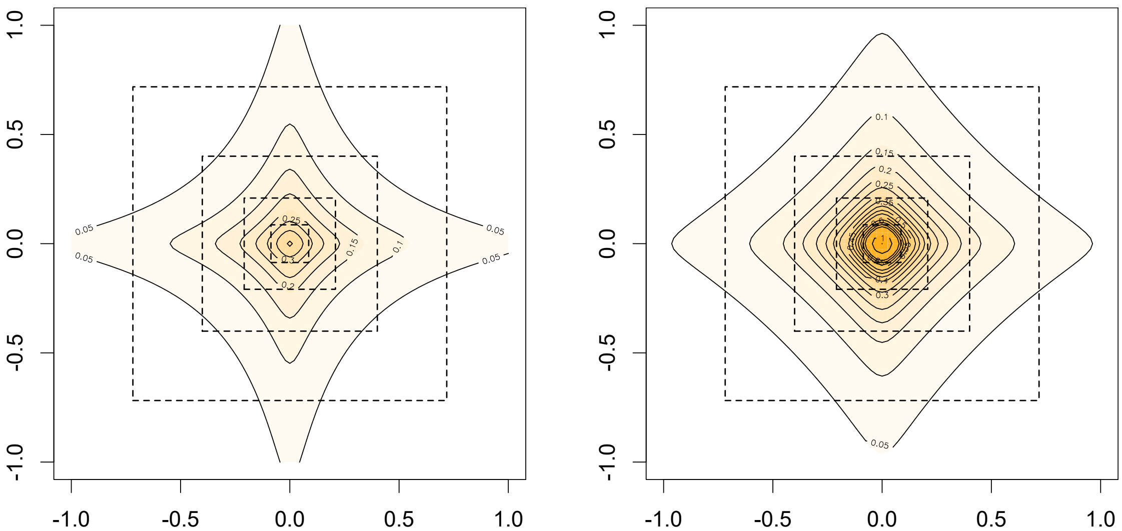

Figure 5: For $\gamma = 1/2$, the density of $P_\alpha \in \mathcal{P}\left(\mathbb{R}^2\right)$ with $\alpha = 1$ (left panel) and $\alpha = 1/2$ (right panel). Several contours of the densities of P_α are given as solid lines. The dashed lines correspond to several contours of the halfspace depth functions of P_α. All these contours take the form of the $\mathcal{L}_1$-balls centered at the origin.

By construction, all such measures $P_\alpha \in \mathcal{P}\left(\mathbb{R}^d\right)$ share identical halfspace depth. An example of two such measures in dimension $d = 2$ is given in Figure 5.

Our main counter-example of paper (C) offers a simple rebuttal to the claim (9) from [37], as the density functions of both measures considered in the example given in Figure 5 are infinitely smooth, but fail to satisfy condition (H) from [37]. As we have seen in the summary of paper (B), condition (H) is essentially a requirement for the existence of Dupin's floating bodies of P, and has nothing to do with the smoothness of the density of P.

Interestingly, our example of explicit halfspace depths for α-symmetric measures applies also for $\alpha > 1$. We obtain that all central regions $D_\delta(P_\alpha)$ are $\mathcal{L}_\beta$-balls centered at the origin with β from (12), and thus they all have smooth boundaries. In particular, all the central regions are Dupin's floating bodies, and by our characterization result presented in paper (A), all α-symmetric distributions with $\alpha > 1$ are uniquely characterized by their halfspace depth.

The same example also illustrates the important limitation of our characterization result from (A). Take the measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ whose characteristic function takes the form (13) with $\gamma = 1/2$ and $\alpha = 1$, and $X \sim P$. Let $u = (u_1, \dots, u_d) \in \mathbb{S}^{d-1}$ be the inner normal of a halfspace $H_{x,u}$ that supports a central region $D_\delta(P)$ for some $\delta \in (0, 1/2)$. Due to the symmetry of the measure P we can consider only the vertex $x = (k, k, \dots, k) \in (0, \infty)^d$ for $k > 0$. Since $H_{x,u}$ must support the $\mathcal{L}_1$-ball

$$D_\delta(P) = \left\{ y \in \mathbb{R}^d \colon \left\| y \right\|_1 \le k \right\},$$

it must be that $u \in [0,\infty)^d \cap \mathbb{S}^{d-1}$. We obtain

$$P(H_{x,u}) = \mathsf{P}\left(\langle X, u\rangle \geq \langle x, u\rangle\right) = \mathsf{P}\left(X_1 \geq \frac{\langle (k,k,\dots,k),(u_1,\dots,u_d)\rangle}{\|u\|_1}\right)$$
$$= 1 - F_1\left(k\frac{\sum_{i=1}^d u_i}{\sum_{i=1}^d |u_i|}\right) = 1 - F_1(k) = \delta,$$

meaning that any supporting halfspace to the central region $D_\delta(P)$ has the same P-mass equal to δ. Necessarily, all Dupin's floating bodies $D_{[\delta]}(P)$ exist for $\delta \in (0,1/2)$. By our result from paper (A), the distribution P is characterized uniquely by its halfspace depth among all distributions that satisfy (7). However, as we demonstrated in our example in Figure 5, it is not true that P is uniquely characterized among all probability measures in $\mathbb{R}^d$; there still exist distributions with the same halfspace depth as P that do not satisfy condition (7).

4 Summary of paper (D): Reconstruction of atomic measures from their halfspace depth

We have seen that only a few types of measures are guaranteed to be characterized by their halfspace depth: (i) univariate measures $P \in \mathcal{P}(\mathbb{R})$, (ii) empirical measures, or slightly more generally, finitely atomic measures in $\mathbb{R}^d$, and (iii) symmetric measures in $\mathbb{R}^d$ whose Dupin's floating bodies exist. In [46], we reconsider finitely atomic measures, and the pioneering results from [98]. For $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ finitely atomic, it is easy to see that there are only finitely many different central regions $D_\delta(P)$ from (3) with $\delta \geq 0$ and that each such a region is a convex polytope.[6] It is this particularly simple structure of the central regions that makes it possible to recover the distribution P from its depth.

The main result from [98] is Theorem 1 which states that any empirical measure $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is uniquely determined by its halfspace depth function. Despite the seemingly simple claim, the proof of this result is rather technical and lengthy; it is first performed for empirical measures P whose atoms $x_1, \dots, x_n$ are in general position in Section 2 (Proposition 1) of [98], and subsequently expanded to the situation when the general position of the atoms might not be valid in Section 3 (Proposition 3). Especially in a situation without a general position, the reconstruction procedure is complicated. It involves the computation of the halfspace depth of every vertex of each polytope $D_{j/n}(P)$, multiple times for all $j = 0, \dots, \lceil n/2 \rceil$, each time w.r.t. a different dataset. The technical proof and the complicated reconstruction procedure make the characterization result somewhat intractable and difficult to approach. Interestingly, [98] is the only paper where explicit reconstruction of an empirical measure from its depth is addressed. The newer articles [44, 36] do not consider this problem.

The main contribution of paper (D) is a substantial simplification of the reconstruction procedure. We do not deal only with empirical measures but consider finitely atomic measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$. Thus, we also allow the atoms of P to take irrational weights.

The first major observation in (D) is Theorem 4, where the reconstruction procedure from [98] is simplified. It is shown that to determine whether a vertex x of a central region $D_\delta(P)$ is an atom, it is enough to compute — only once — a restricted version of the halfspace depth of x, where the halfspace

[6] Formally, this statement is proved as Lemma 1 in (D).

of minimum P-mass is searched for only in the collection of those halfspaces $H \in \mathcal{H}$ that intersect $D_\delta(P)$ exclusively in x. In addition, the P-mass of x is simply the difference between the halfspace depth of x w.r.t. P, and this restricted version of the halfspace depth. Our result is universal in the sense that it is stated for general finitely atomic measures $P \in \mathcal{P}\left(\mathbb{R}^d\right)$, i.e., it holds true whether the atoms of P are in general position or not. This simplifies the procedure from [98].

The second major contribution of (D) is Theorem 8, where it is argued that for all points $x \in \mathbb{R}^d$ that do not lie in the set of halfspace medians of P, it is possible to determine the mass of x directly, without having to compute the halfspace depth at all. Denote by $\delta_0 > 0$ the smallest non-zero depth of a point in $\mathbb{R}^d$ w.r.t. P. The procedure begins with the region $D_{\delta_0}(P)$, which is just the convex hull of the atoms of P. Each vertex x of $D_{\delta_0}(P)$ must be an atom of P, and its probability mass is precisely $D(x;P)$. One proceeds from the lowest levels of the halfspace depth to the higher ones. Suppose all atoms y of P whose halfspace depth $D(y;P) < \delta$ have been identified, together with their P-mass. Take a vertex x of $D_\delta(P)$ such that $D(x;P) = \delta$, and suppose that x is not a halfspace median of P. Then there must exist a face A of the convex polytope

$$U_\delta(P) = \left\{ y \in \mathbb{R}^d \colon D(y;P) > \delta \right\} \subset D_\delta(P)$$

that is "visible" from x, meaning that the open convex hull of x and A does not intersect $U_\delta(P)$. Suppose, for simplicity, that $U_\delta(P)$ is full-dimensional, in which case A can be taken to be a facet (that is, $(d-1)$-dimensional face) of $U_\delta(P)$.[7] The facet A is contained in a unique supporting halfspace $H \in \mathcal{H}$ of the convex polytope $U_\delta(P)$. It turns out that, to determine the P-mass of x, it is enough to shift the halfspace H so that the shifted halfspace $\widetilde{H}$ parallel to H contains x on its boundary. If the shifted halfspace $\widetilde{H}$ contains points of $D_\delta(P)$ other than x, the point x is not an atom of P; if $\widetilde{H}$ supports $D_\delta(P)$ and the only point of intersection of $\widetilde{H}$ and $D_\delta(P)$ is x, then the mass of x is simply $\delta - \delta_x^A$, where δ_x^A is the sum of P-masses of all already known atoms of P that lie inside of $\widetilde{H}$. The main gist of our reconstruction procedure is illustrated in Figure 6.

In our reconstruction procedure in (D), for most vertices x of central regions $D_\delta(P)$, only the probability content of a single halfspace needs to be evaluated to determine the P-mass of x. This stands in sharp contrast with the procedure from [98], where multiple halfspace depths need to be computed to determine the multiplicity of an atom x in the dataset corresponding to P. The paper (D) is concluded with a numerical comparison of the two competing algorithms. As can be seen in Table 1 in (D), the new reconstruction procedure is indeed substantially more efficient than the original program from [98]. For a dataset of $n = 100$ points in $\mathbb{R}^3$, for example, the exact reconstruction of the location and the mass of all data points takes less than 2 seconds using the new procedure. In contrast, the same task takes more than 4 hours of computation time when the original procedure is performed on the same computer.

The main result in (D) states that if $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ is finitely atomic, the position and the weight of all its atoms can be recovered from its depth function. It is, therefore, natural to ask whether it is possible that a measure $P' \in \mathcal{P}\left(\mathbb{R}^d\right)$ that is not finitely atomic can have the same halfspace depth function as a finitely atomic measure. The answer to this question is negative. In **[48]**, we proved that

[7] The procedure described in (D) also works if A is lower-dimensional; we restrict to the most common case of $U_\delta(P)$ being d-dimensional, as the main idea is simpler to understand in this situation; for technical details we refer to Section 2.3 in (D).

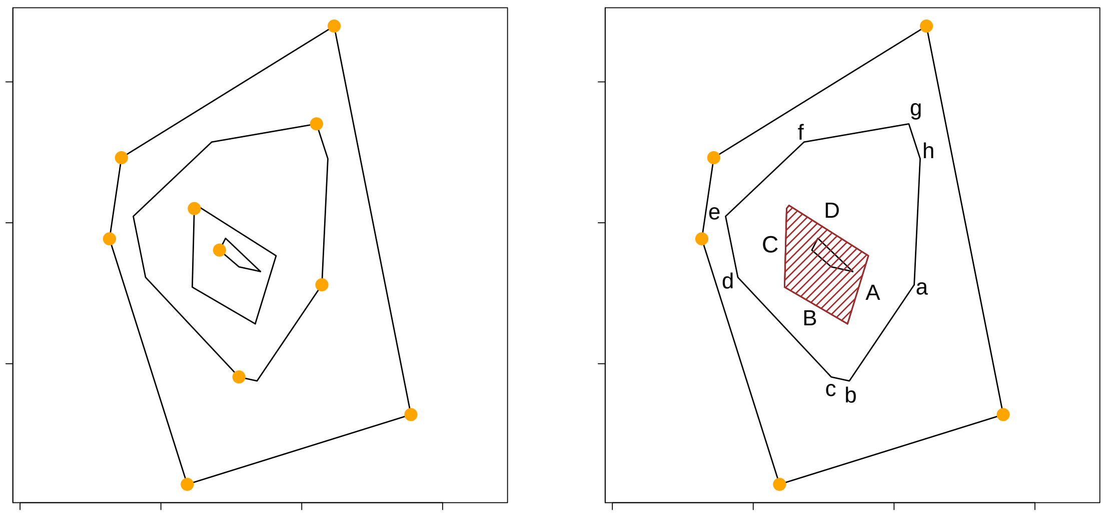

Figure 6: Reconstruction procedure from paper (D). Left: An empirical measure $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ of $n = 10$ points in the plane (orange points) and the complete set of central regions $D_\delta(P)$ (polygons with black boundaries). Right: After a set of five atoms of P has been identified to be the vertices of $D_{1/n}(P)$ (orange points), we intend to reconstruct the location of atoms on the boundary of $D_{2/n}(P)$ (points labeled as a–h). We use $D_{3/n}(P) = U_{2/n}(P)$ (shaded polygon). We consider four facets of $U_{2/n}(P)$ labeled as A–D. Facet A is visible from vertices a, b, and h. Shifting the unique halfplane supporting $U_{2/n}(P)$ at A to a parallel halfplane $\widetilde{H}$ whose boundary passes through b (or h), we see that $\widetilde{H}$ does not support $D_{2/n}(P)$. Thus, neither b nor h is an atom of P. Shifting the same halfplane H to $\widetilde{H}$ with boundary passing through a, we see that $\widetilde{H}$ supports $D_{2/n}(P)$, and contains a single identified atom of P. Thus, $\delta_a^A = 1/n < 2/n = D(a;P)$, and a is an atom of P of weight $D(a;P) - \delta_a^A = 1/n$. Performing the same procedure with, e.g., facet C and vertex e, we see that although the shifted halfplane $\widetilde{H}$ supports $D_{2/n}(P)$, there are two identified atoms of P inside $\widetilde{H}$. Thus, e is not an atom of P because $D(e;P) - \delta_e^C = 2/n - 2/n = 0$.

a measure $P \in \mathcal{P}(\mathbb{R}^d)$ possesses a halfspace depth function that attains only finitely many different values if and only if P is finitely atomic. Therefore, exclusively finitely atomic measures can have depths whose collections of central regions (3) have finitely many extreme points, and only for finitely atomic measures, the reconstruction procedure from paper (D) can be applied directly. In our treatment in (D), we have, therefore, entirely resolved the problem of inverting the halfspace depth function in the situation when the depth attains only finitely many values.

Another follow-up question is the task of recovering the atomic part of a general Borel probability measure $P \in \mathcal{P}(\mathbb{R}^d)$ from its halfspace depth only. While it has been shown in paper (B) that complete reconstruction of the position of the atoms from halfspace depth is impossible, several partial positive results in this direction have been collected in **[47]**. In that contribution, we show that atoms and the general shape of the support of P can be recovered from its halfspace depth under specific conditions.

5 Summary of paper (E): Illumination depth

A major practical obstacle in the applicability of many statistical depths is the difficulty with ties. In data analysis, we do not directly observe the population measure $P \in \mathcal{P}(\mathbb{R}^d)$ that we want to infer about. Instead, we typically have at our disposal only a random sample $X_1, \dots, X_n \in \mathbb{R}^d$ of observations sampled independently from P. Such a random sample of size n corresponds to an empirical measure $P_n \in \mathcal{P}(\mathbb{R}^d)$, which assigns mass $1/n$ to each observation X_i. Naturally, the halfspace depth (5) of P_n attains only finitely many (no more than $\lceil n/2 \rceil$) different values. Even for samples P_n from absolutely continuous measures $P \in \mathcal{P}(\mathbb{R}^d)$, many points in $\mathbb{R}^d$, therefore, must share the same depth value w.r.t. P_n. For example, all points outside the convex hull of the support of P_n (so-called *outsiders*) receive zero depth, no matter how far away from, or how well "fitting" these points are w.r.t. the main bulk of data. This is in the literature frequently called the outsider problem [64, 66, 27, 38]. It renders many depth-ranking methods inefficient and more challenging to work with.

We approached the outsider problem using the geometric concept of *illumination* [103]. The illumination is known to be, in a sense, dual to the floating bodies [61], and by extension, to the halfspace depth. For $K \in \mathcal{K}^d$ a convex body and $\delta > 0$, the δ*-illumination body* is a superset of K given as

$$K^\delta = \left\{ x \in \mathbb{R}^d \colon \text{ volume of the convex hull of } K \cup \{x\} \text{ does not exceed } \operatorname{vol}(K) + \delta \right\}, \qquad (14)$$

where $\operatorname{vol}(\cdot)$ is the usual volume of a convex body. As $\delta \to 0$, the floating bodies "fill in" K from the inside; the illumination bodies approach K from the outside, see Figure 7. That makes illumination amenable to the outsider/ties problem in statistical analysis.

In paper (E) and the follow-up papers **[21, 74]**, we employ illumination for the first time in statistics. We introduce it as a data analysis tool that should be complementary to the halfspace depth or, equivalently, the floating bodies of measures. This combination of the concepts of floating bodies and illumination gives rise to the *illumination depth* of a measure (or a random sample). For $\delta > 0$, a parameter tuning the behavior of the new depth, we propose to rank points $x \in \mathbb{R}^d$ w.r.t. a measure $P \in \mathcal{P}(\mathbb{R}^d)$ based on

- either their halfspace depth $D(x; P)$ if $D(x; P) \geq \delta$,

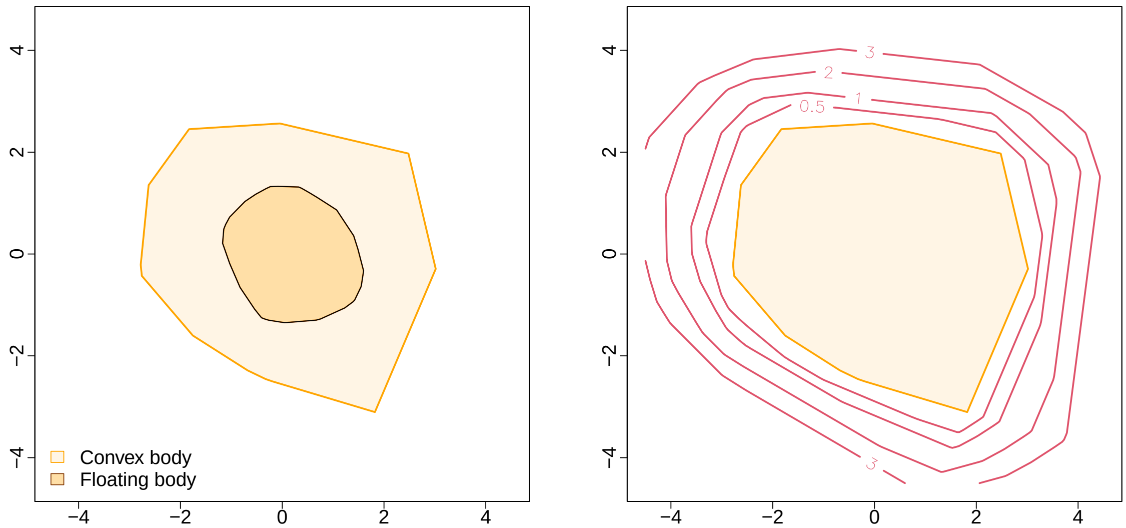

Figure 7: A convex polygon $K \in \mathcal{K}^2$ (in light orange) and its floating body (left panel) K_δ for $\delta = 0.2$ (in darker orange). The floating body coincides with the halfspace depth region $D_\delta(P)$ computed w.r.t. $P \in \mathcal{P}\left(\mathbb{R}^2\right)$ uniform on K. Right panel: Illumination bodies of K for $\delta = 0.5,\ 1,\ 2$ and 3 (bodies with boundaries in red). Illumination bodies are super-sets of K. They are, in a sense, dual to the floating bodies.

- or a decreasing function φ of the illumination $\mathcal{I}(x;P)$ of x onto the central region $D_\delta(P)$

$$\mathcal{I}(x;P) = \frac{\mathrm{vol}\left(\text{convex hull of } x \text{ and } D_\delta(P)\right)}{\mathrm{vol}\left(D_\delta(P)\right)}$$

  in the case when $D(x;P) < \delta$.

The function $\varphi\colon [1,\infty) \to [0,\delta]$ is a given decreasing bijection, which makes the illumination depth approach the value δ as x converges to the boundary of $D_\delta(P)$ from the outside. It also guarantees that as $\|x\| \to \infty$, the illumination depth will decrease to zero. A great deal of attention is devoted to identifying forms of the function φ compatible with the intended statistical applications of the illumination depth; for details, we refer to the main paper (E). For the contours of the illumination onto $D_\delta(P_n)$ for P_n generated by a random sample of points, and two different values of δ, see Figure 8.

The illumination depth presents, in several respects, an improvement over the standard halfspace depth. As we argue in (E), a proper use of the illumination allows us to devise tools that are: (i) conceptually and computationally quite simple; (ii) fully affine invariant; (iii) they have excellent robustness and large sample properties; (iv) are capable of naturally breaking ties in data orderings; (v) can be used for the estimation of extreme quantile regions with efficiency comparable to the state-of-the-art approaches [27, 38]; (vi) are well adjusted to elliptically symmetric distributions; and (vii) are powerful in applications such as classification. An array of theoretical properties of the illumination depth are proved in Section 3 of paper (E), including strong uniform consistency of the sample illumination and the illumination depth as the sample size n goes to infinity (including rates of convergence) and explicit

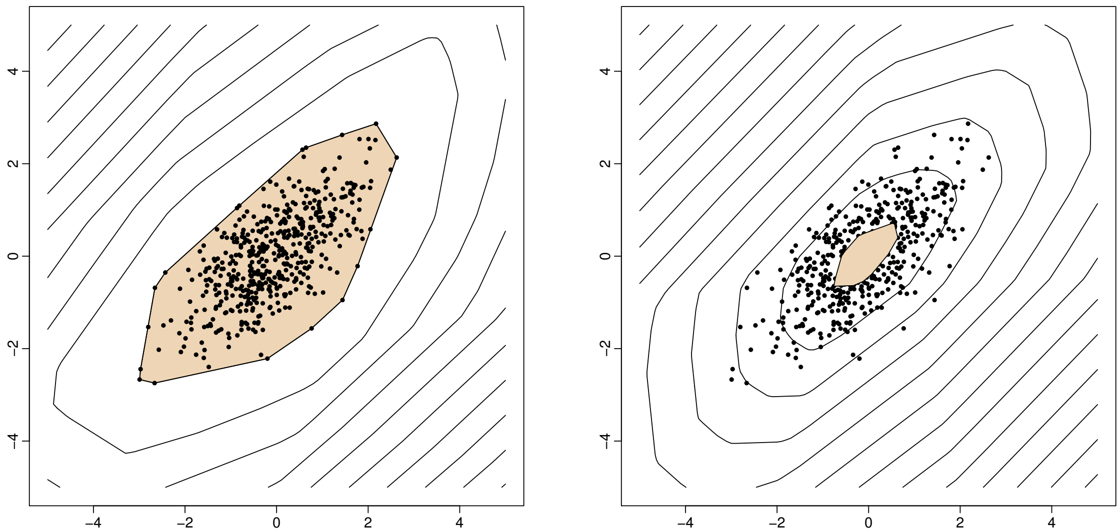

Figure 8: A dataset of size $n = 201$ corresponding to an empirical measure $P \in \mathcal{P}(\mathbb{R}^2)$, and its central region $D_{1/n}(P)$ (left) and $D_{75/n}(P)$ (right). The central region is displayed as the polygon in color. The contours outside the central region represent the boundaries of several illumination bodies of the respective central regions. The illumination adjusts to the covariance structure of the dataset well. This makes it suitable to be used in statistical applications.

expressions for the breakdown point of the illumination. In Section 4 of (E), the illumination is used to obtain refined and robust estimators of the Mahalanobis distance and the usual halfspace depth. Finally, in the last Section 6 of (E), several possible applications of illumination in classical statistical problems are provided. The overall conclusions are quite favorable: proper use of illumination together with the halfspace depth is on par with much more complicated methods that have been proposed in the literature in, e.g., classification [27], or estimation of extreme multivariate quantile regions [38].

6 Summary of paper (F): Uniform convergence rates for the approximated halfspace and projection depth

We have explored the mathematical properties of the halfspace depth in papers (A)–(D) and proposed its application toward data analysis in paper (E). In the last two papers covered in this thesis, we discuss the practical issue of computing the depth $D(x; P)$ of a point $x \in \mathbb{R}^d$ w.r.t. an empirical measure $P \in \mathcal{P}(\mathbb{R}^d)$ of a dataset of a given size. In this section, we write P for the empirical measure and reserve n for the size of the approximation to be used below.

The exact computation of the halfspace depth of a single point in arbitrary dimension $d \geq 2$ is known to be NP-hard [40]. Therefore, a great deal of research has focused on procedures that approximate the true depth [22, 9, 54, 6, 94, 4, 95]. One particularly simple approximation procedure is due to Dyckerhoff [22], who argued that the (multivariate) halfspace depth (5) of a point x can be written as the infimum of (univariate) halfspace depths of projections of x with respect to the projected dataset

$$D(x;P) = \inf_{u\in\mathbb{S}^{d-1}} \min\left\{P_u\left((-\infty,\langle x,u\rangle]\right), P_u\left([\langle x,u\rangle,\infty)\right)\right\}. \tag{15}$$

Here, $P_u \in \mathcal{P}(\mathbb{R})$ stands for the univariate probability measure of the projection $\langle X,u\rangle$ of the random vector $X \sim P$ onto the fixed line in direction $u \in \mathbb{S}^{d-1}$. This suggests the following approximation procedure: (i) draw a random sample of n directions U_i, $i = 1,\ldots,n$, distributed uniformly on the unit sphere $\mathbb{S}^{d-1}$; (ii) evaluate the (univariate) depths of $\langle x,U_i\rangle$ w.r.t. the dataset projected onto U_i for each i; and (iii) approximate the depth (15) of x by the minimum of these numbers. We obtain the *approximated halfspace depth* of $x \in \mathbb{R}^d$ w.r.t. $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ given by

$$D_n(x;P) = \min_{i=1,\ldots,n} \min\left\{P_{U_i}\left((-\infty,\langle x,U_i\rangle]\right), P_{U_i}\left([\langle x,U_i\rangle,\infty)\right)\right\}. \tag{16}$$

Note that the approximated depth $D_n(x;P)$ is a random quantity, as it implicitly depends on the random sample $U_1,\ldots,U_n$ of uniformly distributed directions on the unit sphere. Sometimes, the depth (16) is called the *random halfspace* (or *random Tukey*) depth [9]. Due to its particular simplicity, this depth can actually be found to be the default choice in most resources for the computation of the halfspace depth. For example, for data of dimension $d > 3$, it is implemented in the R computing system [81] in its contributed packages depth [31], depthProc [105], mrfDepth [91], and ddalpha [80].

In paper (F), we provide a theoretical analysis of the random halfspace depth approximation $D_n(x;P)$ from (16). In the main Theorems 1 and 2, we couple the convergence rates of $D_n(x;P)$ toward the actual halfspace depth $D(x;P)$ from (15) with the properties of the distribution $P \in \mathcal{P}\left(\mathbb{R}^d\right)$. In Section 3, we then analyze several scenarios of commonly encountered classes of probability distributions such as (i) distributions with bounded densities; (ii) elliptically symmetric distributions; and (iii) α-symmetric distributions (for definition, see the summary of paper (C) above). In all situations, the same conclusion reappears: The convergence of $D_n(x;P)$ is, especially in dimensions higher than $d > 3$, extremely slow. For quite reasonable distributions P, such as the multivariate normal distributions or alike in dimensions $d > 5$, hundreds of thousands of randomly chosen directions U_i will need to be selected if the halfspace depth of a single point is to be evaluated with reasonable accuracy. Our theoretical findings are supported by an extensive simulation exercise. Consequently, the main conclusion of our paper (F) is quite negative for the random halfspace depth. Our results issue a stark warning about the use of simplistic approximation methods for the computation of halfspace depth for data of higher dimensionality. More sophisticated approximate computation methods are certainly needed for data of dimension $d > 3$.

7 Summary of paper (G): Approximate computation of projection depths

The glum result of our analysis of the random halfspace depth from paper (F) motivated our search for better approximation procedures for the halfspace depth. We did so in the final presented paper (G), where we scoured the literature and adapted eight well known approximation algorithms for the computation of the halfspace depth: • random search, • grid search, • refined random search, • refined grid search, • random simplices, • simulated annealing, • coordinate descent, and • the Nelder–Mead algorithm. A detailed description of all these procedures is given in (G). All these methods are used to numerically minimize the objective function in the definition of the halfspace depth (5)

$$\mathbb{S}^{d-1} \to [0,1] \colon u \mapsto P(H_{x,u})$$

over its domain. Several of the algorithms that we compared are quite simple, and well known to the community. For example, the random search is precisely the random halfspace depth we studied in paper (F). It is expected to perform poorly. Other algorithms, such as the Nelder-Mead procedure [77], had to be adapted to minimization over the unit sphere $\mathbb{S}^{d-1}$ instead of the standardly considered optimization over linear spaces $\mathbb{R}^d$. In paper (G), all eight considered algorithms are adapted to the case of the unit sphere in Section 3. Most of these algorithms involve elaborate tuning procedures for the selection of their parameters. This is done in the first part of Section 4 of (G) in order to perform a fair comparison of all these approaches in an extensive simulation study. In the simulation study, we considered six choices of distributions P: • standard normal distribution, • spherically symmetric Student distribution with five degrees of freedom, • spherically symmetric Student distribution with one degree of freedom (that is, multivariate spherical Cauchy distribution), • the uniform distribution on the unit (hyper-)cube, • a skewed-normal multivariate distribution, and • a multivariate version of the exponential distribution. We performed simulations with random samples of size $n = 10\ 000$, in dimensions $d \in \{5, 10, 15, 20\}$.

The results of our simulation study are reported and discussed in detail in Section 5, and the extensive Supplementary Material **[25]** accompanying paper (G). They are quite telling: Methods based on random projections (such as random search or random simplices) are clearly outperformed by those launching optimization over the surface of the hyper-sphere. Among the latter ones, the (spherical adaptation of the) Nelder-Mead algorithm performs the best, closely followed by coordinate descent and refined random search.

All the considered algorithms have been programmed efficiently in C++ and called from the statistical computation environment R [81]. The methods are currently being prepared to be included in the R contributed package ddalpha [80].

In conclusion, our empirical analysis demonstrates the need for further research on the computational front of depth and multivariate nonparametric statistics. There exist quite promising computational tools applicable also to the task of computation of depth higher dimensions, yet their theoretical study is only in its beginnings. Such advances will be part of our future research.

Conclusions, perspectives, and open problems

We have seen that halfspace depth is a fascinating subject not only in statistics and probability, but also in geometry, mathematical analysis, and computer science. In paper (A) and the follow-up research, we have managed to successfully bring together the halfspace depth, floating bodies, and measures of symmetry. This unique combination allowed us to resolve the long-standing halfspace depth characterization conjecture in paper (C). We did this even though previously it has been, perhaps too optimistically, widely believed that the halfspace depth characterizes all probability distributions (B). On the positive side, we were able to substantially refine a few positive results towards this crucial desideratum of the halfspace depth, especially for measures generated by datasets and, more generally, measures with finite support in (D).

Additional tools of convex geometry, previously unexplored in statistics, have been successfully introduced to the statistical practice in paper (E). Finally, we also addressed the practical problem of the computation of the halfspace depth, especially in the little explored case of higher-dimensional data, which in our setup means data of dimension $d \geq 5$. In paper (F), we provided the first rigorous theoretical analysis of a widely-used approximation scheme and showed that for higher-dimensional data, the results are quite unsatisfactory. In (G) we, therefore, compared several numerical optimization methods that could be used in the computation of halfspace depth. We pointed to a handful of algorithms that, when properly adapted, exhibit superior performance.

Over the years of study of the halfspace depth, we were able to answer many of the questions connected to both its theory and statistical practice. Many more questions, however, still remain. We conclude this thesis by outlining three vital open problems regarding the theoretical properties of the halfspace depth and floating bodies. We argue that they constitute some of the current bottlenecks of the halfspace depth research that hinder many of its immediate applications to multivariate data analysis.

Halfspace depth characterization: Open problems

In paper (C) we answered the general halfspace depth characterization conjecture in the negative. Certain principal closely related questions, however, still remain open: (i) Under what conditions is $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ characterized by its halfspace depth? (ii) How to reconstruct the P-mass of halfspaces that support central regions $D_\delta(P)$ that are not Dupin's floating bodies $D_{[\delta]}(P)$? (iii) If P is uniform on $K \in \mathcal{K}^d$, can we reconstruct K from its single floating body? Based on our research summarized in papers (A)–(D), we conjecture that the halfspace depth determines integrable distributions $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ that satisfy $\int_{\mathbb{R}^d} \|x\| \,\mathrm{d}\, P(x) < \infty$; several partial results point to that class of measures as a suitable candidate. Question (iii) above is of great importance also in geometry and links to the (still open!) *homothety conjecture* — is the ellipsoid the only convex body homothetic to one of its floating bodies [90]?

Smoothness of floating bodies

In paper (A) we have seen that the problem of finding conditions under which the floating body $D_\delta(P)$ has a smooth boundary is important already for Dupin's floating bodies. If $D_\delta(P)$ has a smooth boundary, then $D_{[\delta]}(P)$ exists. Hence, decades of research in that direction exist in geometry [3, 49, 58]. For symmetric log-concave measures, we know that all Dupin's floating bodies exist [3]. Also, uniform distributions on symmetric strictly convex bodies with smooth boundaries possess central regions with smooth boundaries. This resolves a major bulk of this crucial problem — symmetric and very regular measures are bound to have smooth floating bodies.

The geometric results on floating bodies are, however, formulated only for measures satisfying symmetries and strong structural properties. That is not enough for full-blown nonparametric statistical inference. We ask about a generalization of those results to forms suitable for statistical applications: (i) Under what conditions do Dupin's floating bodies exist for asymmetric measures? (ii) Do measures with a smooth, symmetric, strictly quasi-concave density have to possess Dupin's floating bodies? (iii) Connecting this problem with the asymptotic behavior of the depth process (see the summary of paper (A) above), under what conditions is the depth process asymptotically Gaussian? All these questions appear currently within our reach; the geometric techniques applied to log-concave measures are possible to be extended to more general measures. Yet, a precise formulation requires effort and technical derivations. Implications to statistical practice will be rewarding. Having guaranteed regular depth asymptotics at hand, depth-based tests and estimators [23, 67, 84, 102] could be constructed without having to resort to computationally intensive re-sampling techniques, as typically done nowadays.

Floating bodies of polytopes

Another open question asks about the structure of the floating bodies of simple measures. Take $P \in \mathcal{P}\left(\mathbb{R}^d\right)$ uniform on a convex polytope and $\delta > 0$ given. (i) What can be said about the floating body $D_\delta(P)$? (ii) Can its boundary be locally described as a solution to a polynomial equation in d covariates? (iii) If so, what determines the degree of those polynomials, and (iv) how many of such pieces patch the surface of $D_\delta(P)$? All these questions are unexplored, but may be attainable using the current advances in algebraic statistics. It would be quite interesting to completely describe the structure of the boundary of floating bodies of polytopes, and of their algebraic generalizations. Applications would be abundant — uniform measures on simple convex polytopes are among the few whose depth was computed exactly [82]. Their depths are, however, already complicated. Any progress in this direction would provide further examples of floating bodies and exact halfspace depths, so needed for our ultimate understanding of their properties.

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