WDC sets

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Habilitation thesis

1. Introduction

Historically, there have been two classical frameworks, where the notion of curvature was studied and developed. One of them were regular sets with C^2 boundary, where the prinicipal curvatures (i.e. eigenvalues of so-called second fundamental form) are well-defined at each point of the boundary. In this context we define total curvatures $C_k(A)$, $k = 0, \dots d-1$, d being the dimension of the underlying space, as an integral of certain symmetric functions of the principal curvatures. Localizing the integral we then obtain the so-called curvature measures, $C_k(A, \cdot)$. The most important of all curvature measures is the measure $C_0(A, \cdot)$, which allows us to compute the Euler-Poincaré characteristics as

$$\chi(A) = C_0(A) = \int_{\partial A} C_0(A, \cdot).$$

This result is usually referred to as the Gauss-Bonnet formula. The main point is that we are able to compute the global Euler-Poincaré characteristics using the locally determined quantity $C_0(A, \cdot)$.

The second classical framework, where the total curvatures are well-defined, are the convex sets (in $\mathbb{R}^d$). Unlike the smooth sets, the convex sets can have singular parts on the boundary (like corners and edges) and the principal curvatures may not be well-defined at each point of the boundary. A completely different approach is then needed and, for a nonempty compact convex set $K \subset \mathbb{R}^d$, the total curvatures are defined via the so-called Steiner formula

$$\text{volume}(\{x \in \mathbb{R}^d : \text{dist}\,(x, K) \le \varepsilon\}) = \sum_{k=0}^{d} C_k(K)\varepsilon^{d-k}, \quad \varepsilon > 0,$$

which tells us that the volume of an ε-neighbourhood of a convex can be expressed as a polynomial of degree d and the total curvatures are (after renormalization) the coefficients of that polynomial. Here the 0-th total curvature (i.e. the coefficient by ε^d, after renormalization) is the same for every convex set, which is basically a trivial version of the Gauss-Bonnet theorem (the Euler characteristics of a nonempty convex set is always 1). The 1-st total curvature (the coefficient corresponding to ε^{d-1}) can be interpreted as a mean width of the set K, the linear term corresponds to the area of ∂K and the absolut term $C_d(K)$ is simply the volume of K. The curvature measures $C_k(K, \cdot)$ are similarly defined by a localized version of the Steiner formula. It is a remarkable fact that, for smooth convex sets, those two completely different approaches yield the same curvature measures.

Those two rather different historical frameworks were unified by H. Federer in 1959 in his seminal paper [1], which is considered the begining of the modern curvature theory. The unifying framework, the sets of positive reach, combines topological flexibility of the smooth sets and a possible singular behaviour of the convex sets.

A set $A \subset \mathbb{R}^d$ is said to be of a positive reach if there is some $\varepsilon_0 > 0$ such that every x satisfying $\text{dist}\,(x, A) \le \varepsilon$ has the unique nearest point in A. The total curvatures and curvature measures are then again defined via the Steiner formula, which can (in general) no longer hold for every $\varepsilon > 0$, but will still hold if $\varepsilon > 0$ is small enough.

The key point why the total curvatures make sense in this more general setting is that the sets of positive reach can only have outward corners and therefore the singular part of the boundary can only carry a positive curvature and the negative curvature can be nicely controlled. Both compact convex sets and compact smooth sets (i.e. sets with C^2 boundary) are of positive reach. Apart from the Gauss-Bonnet theorem, these curvature measures also satisfy (as proved by Federer) the so-called Principal Kinematic Formula

$$\int_{\mathcal{G}_d} C_k(A \cap gB)\, dg = \sum_{i+j=d+k} \gamma_{d,i,j} C_i(A) C_j(B).$$

Here, on the left hand side, we integrate the k-th curvature measure of the intersection of A and gB, where g runs through all possible rigid movements $\mathcal{G}_d$ and the integration is with respect to the unique Haar measure on $\mathcal{G}_d$. Moreover, $\gamma_{d,i,j}$ are constants depending only on d, i and j. In particular, the intersection of two sets of positive reach in general position is again a set of positive reach.

In 1986 M. Zähle discovered (in [15]) that the curvature measures for the sets of positive reach admit certain current representation (in the sense of the current theory by Federer and Flemming) and three years later (in [2]) J. Fu proved that the curvature measures of a set A can be derived from the so-called normal cycle of A (usually denoted as N_A), which is an integral current (again in the sense of the Federer-Flemming theory of currents) lying in $\mathbb{R}^d \times \mathbb{S}^{d-1}$.

In 1994 J. Fu developed a new method for construction of the normal cycle (and therefore the existence of the curvature measures) and used it to prove the existence of the normal cycle (as well as the Gauss-Bonnet formula and the Principal Kinematic Formula for the corresponding curvature measures) for the class of subanalytic sets. As the key concepts, his method (which also plays a prominent role in the context of this thesis) uses the notions of Monge-Ampère functions and weakly regular values.

The Monge-Ampère functions (defined on an open set $U \subset \mathbb{R}^d$) are characterized by the existence of a specific integral current on $U \times \mathbb{R}^d$ (similarly to the normal cycle), which, for a function f, is denoted df and, roughly speaking, represents the integration over the graph of the derivative of f. In practice, however, it is usually enough to work with the following much more simple condition, which defines the so-called strongly approximable functions: f is strongly approximable on an open set U if there is a sequence $f_1, f_2, \ldots \in C^2(U)$ that converges to f in $L^1_{\mathrm{loc}}(U)$ and such that

$$\int_K \left| \det \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{i \in I, j \in J} \right| \le C(f, K) \tag{1.1}$$

for every $K \subset U$ compact and $I, J \subset \{1, \ldots, d\}$ of the same cardinality, here $C(f, K)$ is a constant depending only on f and K. Each strongly approximable function is Monge-Ampère and although it is not known whether the converse is true, all known sub-classes of the Monge-Ampère functions (like the subanalytic functions, the Sobolev spaces $W_2^n(\mathbb{R}^n)$, semiconvex, or delta convex functions) consist of strongly approximable functions.

For a Lipschitz function f, a real number c is called a weakly regular value if there is an $\varepsilon > 0$ such that the inequality $|v| \ge \varepsilon$ holds whenever $c < f(x) < c + \varepsilon$ and $v \in \partial f(x)$. Here $\partial f(x)$ denotes the usual Clarke subgradient on f at x.

The procedure of constructing the normal cycle of a set A goes as follows: first we need to find a Monge-Ampère function f such that $f^{-1}((-\infty, 0]) = A$ and that 0 is a weakly regular value of f. Such a function f is called an aura for the set A. In the next step, we project the current df to $A \times \mathbb{S}^{d-1}$ to obtain the normal cycle N_A.

Often (but not always) there is a natural choice of an aura for a set A. For instance, if A is a set of positive reach, the distance function $f(x) = \operatorname{dist}(x, A)$ always works.

Over the following years, J. Rataj and M. Zähle proved the existence of the normal cycle for two other classes of sets, the so-called $\mathcal{U}_{\mathrm{PR}}$ sets (in [12]) and also the Lipschitz manifolds of bounded inner curvatures (in [13]).

One more class was considered as a possible candidate for admitting curvature measures and that were the delta convex surfaces. A function is called delta convex if it can be expressed as a difference of two convex functions. The key point is that although the delta convex functions can have (unlike semiconvex functions) singular parts pointing upwards and downwards (i.e. the delta convex surfaces can have both outwards pointing corners and inwards pointing corners, which is forbidden for sets of positive reach).

The first result in this direction of research was a 2-dimensional case resolved in 2000 by J. Fu. In general dimension, the case of delta convex functions has been resolved by D. Pokorný and J. Rataj in a surprising manner, after they discovered the following determinant formula

$$\det(A - B) = \frac{1}{d!} \sum_{k=0}^{d} (-1)^k \binom{d}{k} \det((d-k)A + kB).$$

The key feature of the formula is that while the left hand side contains a determinant of the difference of two matrices A and B, the right hand side consists of determinants of only positive linear combinations of A and B. This essentially allows us to reduce the integrability of the Jacobians of the delta convex functions in (1.1) to the integrability of the Jacobians of certain convex functions, which is a simple task. This way we can prove that yhe delta convex functions are strongly approximable and therefore Monge-Ampère.

This led, applying the previously mentioned method of Fu, to the new class of sets admitting the curvature measures, the so-called WDC sets (sublevel sets of the delta convex functions at weakly regular values), which, in particular, generalize the theory of the sets of positive reach.

This thesis consists of five papers the aim of which is to develop various aspects of this new class of sets admitting the classical results of integral geometry. The particular papers are the following:

- J.H.G. Fu, D. Pokorný, J. Rataj: Kinematic formulas for sets defined by differences of convex functions. *Adv. Math. 311* (2017), 796–832,
- D. Pokorný, J. Rataj, L. Zajíček: On the structure of WDC sets. *Math. Nachr.* 292 (2019), 1595–1626,
- D. Pokorný, L. Zajíček: On sets in $\mathbb{R}^d$ with DC distance function. *J. Math. Anal. Appl.* 482 (2020), No. 1.
- D. Pokorný, L. Zajíček: Remarks on WDC sets. To appear in *Comment. Math. Univ. Carolin.*
- D. Pokorný: Curvature measures for unions of WDC sets. Submitted.

The results of [7] were vastly generalized by J. Fu, D. Pokorný and J. Rataj in [6] which is the first part of this thesis. The paper contains two main results. First one is a generalization of the classical result by Evald, Larman and Rogers and reads as follows:

Theorem 1.1. *Let $K \subset \mathbb{R}^d$ be closed and convex. Denote by T_K the set of pairs $(v, w) \in S^{d-1} \times S^{d-1}$ with the property that there exists a nondegenerate segment $\tau \subset \partial K$ with direction v and lying in a supporting hyperplane of K with outward normal direction w. Then T_K has σ-finite $(d-2)$-dimensional Minkowski content.*

Even though the above theorem is interesting in its own merits, for our main goal it is mostly a technical result that has the following corollary:

Corollary 1.2. *If $f : \mathbb{R}^d \to \mathbb{R}$ is DC then* $\operatorname{graph} \partial f$ *has σ-finite d-dimensional Minkowski content.*

This, finally, is the main ingredient for the main goal of the paper, which is the proof of the kinematic formula for pairs of the WDC sets:

Theorem 1.3. *Let $A, B \subset \mathbb{R}^d$ compact WDC sets and let $0 \leq k \leq d$. Then*

$$\int_{\mathcal{G}_d} C_k(A \cap gB)\, dg = \sum_{i+j=d+k} \gamma_{d,i,j} C_i(A) C_j(B).$$

Here $\gamma_{d,i,j}$ are constants depending only on d, i and j.

Even though the WDC sets have a relatively simple definition it is not clear whether they allow any simple geometric description (which is in fact also true for the sets of positive reach). The second part of the thesis, which is a paper by D. Pokorný, J. Rataj and L. Zajíček [8], is therefore devoted to the study of the structure of the WDC sets.

A set $A \subset \mathbb{R}^d$ will be called a Lipshitz (DC) manifold if it can be locally expressed as an isometric copy of the graph of a Lipschitz (DC) mapping from $\mathbb{R}^k$ to $\mathbb{R}^{d-k}$. Similarly, a Lipshitz (DC) domain is a set that can be locally expressed as an isometric copy of the subgraph of a Lipschitz (DC) function. The main non-technical results of the paper are the following:

Theorem 1.4. *For each closed locally WDC set $M \subset \mathbb{R}^d$, its boundary ∂M can be locally covered by finitely many DC hypersurfaces.*

Theorem 1.5. *Let a closed set $A \subset \mathbb{R}^d$ be a Lipschitz manifold of dimension $0 < k < d$. Then A is a locally WDC set if and only if A is a DC manifold of dimension k.*

Theorem 1.6. *Let $A \subset \mathbb{R}^d$ be a closed Lipschitz domain. Then A is locally WDC if and only if A is a closed DC domain.*

When working with the WDC sets there is often one technical obstacle: we don't know how to construct an aura for a given WDC set M. Recall that, for the sets of positive reach, which form a subclass of the WDC sets, the distance function is always an aura. The study of the properties of the WDC sets has been therefore continued later in two papers [9] and [10] by D. Pokorný a L. Zajíček which are dedicated to the following question: is the distance function $x \mapsto \operatorname{dist}(x, M)$ always an aura for any WDC set M?

The main point proven in [9] is

Theorem 1.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a DC function. Then the distance function* $d := \operatorname{dist}(\cdot, \operatorname{graph} f)$ *is DC on $\mathbb{R}^2$.*

This leads to the result from [10]

Theorem 1.8. *Let $M \neq \varnothing$ be a closed locally WDC set in $\mathbb{R}^2$. Then the distance function d_M is a DC aura for M. In particular, M is a WDC set.*

Note that the last part of the theorem tells us that the locally WDC sets and the WDC sets are the same class in $\mathbb{R}^2$, which is still unknown for the higher dimensions.

The key consequence is that we now have a natural choice of an aura for the WDC sets in $\mathbb{R}^2$, which for instance can be used to prove the following:

Theorem 1.9. *The class of all nonempty compact WDC sets in $\mathbb{R}^2$ is an $F_{\sigma\delta\sigma}$ subset of the metric space of nonempty compact WDC sets in $\mathbb{R}^2$ (equipped with the usual Hausdorff metric).*

The importance of this result is the fact that it suggests that (at least in $\mathbb{R}^2$) a theory of point processes on the space of the compact WDC sets (analogous to the concept of point processes on the space of sets of positive reach introduced in [14]) can be build.

The goal of the last section, which corresponds to the paper by D. Pokorný [11], is to push the curvature theory of the WDC sets a little bit further. Its purpose is twofold: first to develop the curvature theory for $\mathcal{U}_{WDC}$ sets similar to the theory of $\mathcal{U}_{PR}$ sets developed by J. Rataj and M. Zähle, and second to obtain a natural candidate for a maximal integral geometric regularity class (in the sense of [6] and [5]) in dimension 2.

A set $M \subset \mathbb{R}^d$ is a $\mathcal{U}_{WDC}$ set if for every $x \in M$ there is a neighbourhood U of x and sets $M_1, \dots, M_j$ such that $M \cap U = U \cap \bigcup_{i=1}^{j} M_i$ and such that each set $\bigcap\limits_{i \in I} M_i$, $I \in \Sigma_j$, is WDC. The main results of this section are:

Theorem 1.10. *Each compact $\mathcal{U}_{WDC}$ set admits the normal cycle.*

and the kinematic formula for pairs of the WDC sets:

Theorem 1.11. *Let M and K be two compact $\mathcal{U}_{WDC}$ sets in $\mathbb{R}^d$ and let $0 \le k \le d-1$. Then $M \cap g(K)$ is WDC for almost every $g \in \mathcal{G}_d$ and*

$$\int_{\mathcal{G}_d} C_k(M \cap gK, U \cap gV)\, dg = \sum_{i+j=d+k} \gamma_{d,i,j} C_i(M,U) C_j(K,V), \tag{1.2}$$

where $\gamma_{d,i,j}$ are constants depending only on d, i and j.

Also we prove the following geometric characterisation of the WDC sets in dimension 2:

Theorem 1.12. *Let $M \subset \mathbb{R}^2$ be a compact set. Then the following conditions are equivalent.*

(1) M is $\mathcal{U}_{WDC}$,
(2) M^c has finitely many connected components and ∂M is a union of finitely many DC graphs.

Here by a DC graph we mean a set that is an isometric copy of a graph of a DC function over an interval. Moreover, we prove that in dimension 2 all known classes of compact sets that admit the normal cycle are contained in the class of the WDC sets. This and othere results mentioned in the paper suggest that the class of $\mathcal{U}_{WDC}$ sets is a natural candidate for a maximal integral geometric regularity class in $\mathbb{R}^2$ (in the sense of Fu [5]).

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KINEMATIC FORMULAS FOR SETS DEFINED BY DIFFERENCES OF CONVEX FUNCTIONS

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On sets in $\mathbb{R}^d$ with DC distance function

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Remarks on WDC sets

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