Mathematical analysis of nonlinear systems describing flows of incompressible fluids

(Habilitation Thesis)

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Matematická anlýza nelineárních systémů popisujících proudění nestlačitelných tekutin

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Contents

Structure of the thesis

The habilitation thesis is based on the following articles:

[B1] M. Bulíček, P. Gwiazda, J. Málek and A. Świerczewska-Gwiazda: On unsteady flows of implicitly constituted incompressible fluids, *SIAM J. Math. Anal.*, **44**(4): 2756–2801, 2012
DOI: 10.1137/110830289

[B2] M. Bulíček, F. Ettwein, P. Kaplický, D. Pražák. On uniqueness and time regularity of flows of power-law-like non-Newtonian fluids. *Math. Methods Appl. Sci.* **33**:1995–2010, 2010
DOI: 10.1002/mma.1314

[B3] M. Bulíček, P. Kaplický and J. Málek: An L2-maximal regularity result for the evolutionary Stokes–Fourier system, *Appl. Anal.*, **90**(1): 31–45, 2011
DOI: 10.1080/00036811003735931

[B4] M. Bulíček, P. Kaplický, and D. Pražák: Time regularity of flows of non-Newtonian fluids with critical power-law growth, accepted to *Mathematical Models and Methods in Applied Sciences*, arXiv:1802.10053, 2019
DOI: 10.1142/S0218202519500209

[B5] M. Bulíček and J. Málek: Large data analysis for Kolmogorov's two-equation model of turbulence, *Nonlinear Analysis: Real World Applications*, **50**:104–143, 2019
DOI: 10.1016/j.nonrwa.2019.04.008

[B6] M. Bulíček, J. Málek and T. N. Shilkin: On the Regularity of Two-Dimensional Unsteady Flows of Heat-Conducting Generalized Newtonian Fluids, *Nonlinear Analysis: Real World Applications*, **19**: 89–104, 2014
DOI: 10.1016/j.nonrwa.2014.03.003

General characterization of the above list of papers is that they focus on the existence and the qualitative analysis of solutions to certain systems of partial differential equations (PDEs) that arise in modelling of unsteady flow of incompressible homogeneous fluids with complicated rheology. We consider three general classes of such fluids: purely mechanical setting, heat conducting fluids and the model for turbulence. In the purely mechanical setting, the starting point for the further analysis is the paper [**B1**], where the existence of a weak solution is established for, up to date, most general classes of fluids, where the Cauchy stress and the symmetric part of the velocity gradient are related through an implicit algebraic law. The uniqueness of a weak solution in sense of trajectories is proved in [**B2**] and the uniqueness and global in time regularity estimates are proven in [**B4**] for models, where we have the validity of the energy equality. These purely mechanical models can be then also naturally extended to the full thermodynamical setting and the papers [**B3**, **B6**] deals with the regularity of solution to these generalized models. Finally, since the models of heat conducting fluids and the models of turbulence share the very similar properties and difficulties, we present the first existence result for the Kolmogorov two equation model of turbulence in the thesis, which was established in [**B5**].

1 Homogeneous incompressible fluids

System of partial differential equations appearing in the continuum thermodynamics is a consequence of balance equations for mass, angular and linear momentum, energy and possibly other quantities we are interested in and the system is required to satisfy the entropy inequality (the second law of thermodynamics). This system contains many physical quantities, e.g., the density, the velocity, the internal energy, the heat flux, the Cauchy stress, which are in addition required to fulfill the constitutive relations that characterize the response of the material to external stimuli. In case of incompressible homogeneous fluids, which is the topic of the thesis, the Cauchy stress is usually related to the velocity gradient through its symmetric part and the heat flux to the temperature gradient. In addition, these constitutive relations may be also affected by the dependence on other physically relevant quantities.

Considering the purely mechanical setting, i.e., the flow under constant temperature, the system of equations reduces to

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{and} \qquad \varrho \left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) - \operatorname{div} \mathsf{S} = -\nabla p + \varrho \mathbf{b}\,, \tag{1.1}$$

which is supposed to be satisfied in space-time cylinder $Q := (0,T)\times\Omega$, where $\Omega \subset \mathbb{R}^3$ (sometimes we shall consider also the two dimensional setting) and $T > 0$ is the length of time interest. Here, $\varrho \in (0,\infty)$ is the constant density, which we will set to be equal to one in the rest of the thesis, $\mathbf{b}$ is the density of the external forces, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity, p is the mean normal stress and S, a part of the Cauchy stress $\mathsf{T} = -p\mathsf{I} + \mathsf{S}$, is the only quantity that characterizes material properties of a given fluid. Assuming that the fluid is no-polar, the balance of angular momentum dictates that S is symmetric.

Due to the second law of thermodynamics, even in the isothermal setting we have the following restriction (the equality holds thanks to $\operatorname{div} \boldsymbol{v} = 0$)

$$\mathsf{T} \cdot \mathsf{D} = \mathsf{S} \cdot \mathsf{D} \geq 0\,, \tag{1.2}$$

where $\mathsf{D} = \mathsf{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ is the symmetric part of the velocity gradient. On the other hand, the quantity $\mathsf{S} \cdot \mathsf{D}$ appears[1] also in the balance

[1] We denote here $\mathsf{A} \cdot \mathsf{B}$ the scalar product of two matrices. Also in what follows we use this symbol for a general notion of scalar product for vector- and for tensor-valued functions as well.

of the kinetic energy. Indeed, taking the scalar product of $(1.1)_2$ and $\boldsymbol{v}$, we obtain the identity (notice, we set $\varrho := 1$)

$$\partial_t \left(\tfrac{1}{2}|\boldsymbol{v}|^2\right) + \operatorname{div}\left((p + \tfrac{1}{2}|\boldsymbol{v}|^2)\boldsymbol{v}\right) - \operatorname{div}(\mathsf{S}\boldsymbol{v}) + \mathsf{S}\cdot\mathsf{D} = \mathbf{b}\cdot\boldsymbol{v}\,. \tag{1.3}$$

The above identity integrated over Ω leads after integration by parts to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{v}|^2\,dx + \int_\Omega \mathsf{S}\cdot\mathsf{D}\,dx = \int_\Omega \mathbf{b}\cdot\boldsymbol{v}\,dx - I_{\partial\Omega}\,, \tag{1.4}$$

where

$$I_{\partial\Omega} := \int_{\partial\Omega} \left((p + \tfrac{1}{2}|\boldsymbol{v}|^2)\boldsymbol{v}\cdot\boldsymbol{n} - \mathsf{S}\boldsymbol{v}\cdot\boldsymbol{n}\right)\,dS \tag{1.5}$$

and $\boldsymbol{n} = \boldsymbol{n}(x)$ denotes an outer normal vector to $\partial\Omega$ at $x \in \partial\Omega$. Thus, if boundary conditions are set such that $I_{\partial\Omega} \geq 0$, then (1.4) together with (1.2) gives the natural energy estimate.

In this thesis, we consider exactly such boundary conditions for which the term $I_{\partial\Omega}$ is nonnegative. First, it is evident, that if we consider impermeable boundary, i.e.,

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } (0,T)\times\partial\Omega\,, \tag{1.6}$$

then the formula for $I_{\partial\Omega}$ reduces to

$$I_{\partial\Omega} := -\int_{\partial\Omega} \mathsf{S}\boldsymbol{v}\cdot\boldsymbol{n}\,dS = -\int_{\partial\Omega} \mathsf{S}\boldsymbol{n}\cdot\boldsymbol{v}\,dS\,, \tag{1.7}$$

where for the second equality we used the symmetry of S. Finally, to guarantee the sign, we shall assume the Navier slip boundary conditions

$$(\mathsf{S}\boldsymbol{n})_\tau = -\gamma_*\boldsymbol{v}_\tau \quad \text{on } (0,T)\times\partial\Omega\,, \tag{1.8}$$

where we denoted $\boldsymbol{z}_\tau := \boldsymbol{z} - (\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{n}$, the projection of a vector $\boldsymbol{z}$ defined at $x \in \partial\Omega$ to the tangent plane located at x. For $\gamma_* \in (0,\infty)$, the condition (1.8) is referred as the Navier slip condition, for $\gamma_* = 0$, it reduces to the perfect slip boundary conditions and the value $\gamma_* = \infty$ formally reduces to the no-slip boundary conditions, i.e.,

$$\boldsymbol{v} = \mathbf{0} \text{ on } (0,T)\times\Omega\,. \tag{1.9}$$

Thus, if we complete the considered problem by formulating the initial condition

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega, \tag{1.10}$$

where $\boldsymbol{v}_0$ is a given function fulfilling the compatibility conditions $\operatorname{div} \boldsymbol{v}_0 = 0$ in Ω and $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, and if we assume that the initial velocity $\boldsymbol{v}_0$ and $\mathbf{b}$ are given L^2-integrable functions then (1.4) implies that

$$\sup_{t\in[0,T]} \int_\Omega |\boldsymbol{v}|^2 \, dx + \int_0^T \int_\Omega \mathsf{S} \cdot \mathsf{D} \, dx \, dt + \gamma_* \int_0^T \int_{\partial\Omega} |\boldsymbol{v}|^2 \, dS \; dt < \infty \, . \tag{1.11}$$

Thus, we see that under the assumption (1.2) and for naturally integrable data, we obtained a priori long-time estimates. From the mathematical point of view it is therefore desirable to address the questions

Q1 For how much general class of relations between S and D fulfilling (1.2) can we obtain the global in time existence of a **weak solution** for large data?

Q2 For how much general class of relations between S and D fulfilling (1.2) and for which data can we obtain the uniqueness (and regularity) of a **weak solution**?

In what follows, we want to answer these two questions for models, where S and D are related through the implicit relation (here $\mathsf{G} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is continuous mapping)

$$\mathsf{G}(\mathsf{D}, \mathsf{S}) = \mathbf{0} \, , \tag{1.12}$$

which still guarantees (1.2).

Also on purpose, we want to deal with the notion of a **weak solution**, which means that (1.1) is satisfied in the sense of distributions, since this concept seems to be very natural for continuum mechanics, see [74, 18]. Moreover, in the simplest case when $\mathsf{S} = 2\nu\mathsf{D}$ with ν being a constant, the problem (1.1) reduces to the Navier–Stokes equations, for which the existence of global in time weak solution has been established long time ago (see [52] on the whole space and [41] for bounded domain) but the global in time uniqueness or regularity of the solution is one of the most famous open problems in mathematics.

We complete this introductory part by considering a more difficult class of problems, which is the case when S and D still satisfy (1.2) but S is not related only to $\mathbf{D}$ by the implicit relation (1.12) but also to other unknowns (possibly vector valued) z through

$$\mathbf{G}(z, \mathbf{D}, \mathbf{S}) = \mathbf{0}\,, \tag{1.13}$$

where z satisfies an equation

$$\partial_t z + \operatorname{div}(\boldsymbol{v} z) - \operatorname{div} \boldsymbol{q} = F \text{ in } Q \tag{1.14}$$

and fulfils the initial and the boundary data[2]

$$z(0) = z_0 \text{ in } \Omega, \qquad z = z_d \text{ on } (0,T) \times \partial\Omega. \tag{1.15}$$

We have in mind mainly two cases. First, when z represents the internal energy e and the second, when $z := (b, \omega)$ represents the turbulent kinetic energy and frequency. This means, we consider the following two cases

$$z = \begin{cases} e \quad \text{then} & \boldsymbol{q} = \kappa(e, |\nabla e|)\nabla e, \quad F = \mathbf{S} \cdot \mathbf{D}, \\ (b,\omega) \quad \text{then} & \boldsymbol{q} = \dfrac{b}{\omega}\nabla(b,\omega), \quad F = \tilde{F}(b, \omega, \nabla \boldsymbol{v}). \end{cases} \tag{1.16}$$

In the first case, (1.14) is just the balance of internal energy,[3] while in the second case, (1.14) forms two convection–diffusion equations for b and for ω, see **[B5]** and references therein.

[2]The Dirichlet boundary condition can be also replaced by the Neumann boundary condition

$$\boldsymbol{q} \cdot \boldsymbol{n} = 0 \text{ on } (0,T) \times \Omega.$$

[3]If we assume that internal energy is an invertible function of temperature, then $\boldsymbol{q}$ is just a generalized heat flux.

2 Implicitly constituted incompressible fluids

In this part, we restrict ourselves to the purely mechanical setting and do not consider any dependence of the constitutive equations on the unknown z and present the results obtained for such models in [**B1**].

2.1 Constitutive relations

The most studied model for incompressible fluids is the case when

$$\mathbf{S} = \tilde{\mathbf{S}}(\mathbf{D}) \tag{2.1}$$

with some $\tilde{\mathbf{S}} : \mathbb{R}^3 \to \mathbb{R}^3$. It is a generalization of the fundamental Newton's statement [73], which can be transferred to the relation

$$\mathbf{S} = 2\nu_* \mathbf{D} \qquad \nu_* \in (0, \infty)\,. \tag{2.2}$$

However, the Newton model (2.2) or the generalized model (2.1) are not capable to describe and/or to explain many physically relevant phenomena. Therefore in the original works of Rajagopal [77, 78] and Rajagopal and Srinivasa [79], the authors investigated systematically the models of the form

$$\mathbf{G}(\mathbf{D}, \mathbf{S}) = \mathbf{0}\,,$$

or even more generally (recall that $\mathbf{T}$ denotes the Cauchy stress tensor)

$$\tilde{\mathbf{G}}(\mathbf{D}, \mathbf{T}) = \mathbf{0}\,. \tag{2.3}$$

Obviously, in comparison with traditional models of the class (2.1), the implicit equations (1.12) or (2.3) can describe much more complicated responses. The class (2.3) is capable of capturing several non-Newtonian phenomena such as shear-thinning, shear-thickening, the presence of an activation criteria, the yield stress phenomena etc. The class (2.3) can even justify and explain the pressure thickening, i.e., the pressure dependent viscosity in incompressible fluids[4].

[4]Such models are important in many applications. The fact that viscosity should depend on the pressure has been already proposed by Stokes [86] and confirmed experimentally by Barus [5], see also book by Bridgman [14]. However, this class of incompressible materials that fits to implicitly constituted fluids (2.3) is not investigated in this study. We refer to [17] for the most recent results concerning mathematical analysis of incompressible fluids with the pressure and the shear rate dependent viscosity. Moreover, the above classes of fluids can be still justified from the thermodynamical point of view, see [79] and [70].

We recall here one prominent example that can be described by (1.12). Let us consider the equation (here ν can be understood as a generalized viscosity and τ_+ as the yield stress)

$$2\nu(|\mathbf{D}|)\,(\tau_* + (|\mathbf{S}| - \tau_*)_+)\;\mathbf{D} = (|\mathbf{S}| - \tau_*)_+\,\mathbf{S} \quad \text{with } \tau_* > 0\,, \tag{2.4}$$

where a_+ denotes the positive part of a, i.e., $a_+ = \max\{a, 0\}$. Setting

$$\mathbf{G}(\mathbf{D}, \mathbf{S}) = 2\nu(|\mathbf{D}|)\,(\tau_* + (|\mathbf{S}| - \tau_*)_+)\;\mathbf{D} - (|\mathbf{S}| - \tau_*)_+\,\mathbf{S}, \tag{2.5}$$

we see that (2.4) is of the form (1.12). More importantly, (2.4) is equivalent to the classical description of fluids of a Bingham or a Herschel–Bulkley type (see [29]):

$$|\mathbf{S}| \le \tau_* \Leftrightarrow \mathbf{D} = \mathbf{0} \quad \text{and} \quad |\mathbf{S}| > \tau_* \Leftrightarrow \mathbf{S} = \frac{\tau_* \mathbf{D}}{|\mathbf{D}|} + 2\nu(|\mathbf{D}|)\mathbf{D}. \tag{2.6}$$

Model (2.4) can cover (by setting $\tau_* = 0$) the classical models of fluids with shear dependent viscosity (see (2.1))

$$\mathbf{S} = 2\nu(|\mathbf{D}|)\mathbf{D} \quad \text{with} \quad \nu : \mathbb{R}^+ \to \mathbb{R}^+\,, \tag{2.7}$$

including the classical Ladyzhenskaya model

$$\mathbf{S} = 2\nu_*(\alpha_* + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D} \quad \text{with} \quad r \ge 1,\ \nu_* > 0,\ \alpha_* \in [0, \infty). \tag{2.8}$$

The power-law model is then achieved by setting $\alpha_* = 0$ and the classical Navier-Stokes model (2.2) is achieved by taking $r = 2$ in (2.8).

From the mathematical point of view, the class of implicitly constituted fluids is indeed attracting. While, the classical form (2.6), in which the response of fluids with the yield stress or the activation criterion is often written, gave birth to use for the mathematical analysis such tools as variational inequalities, multi-valued function analysis, functions with discontinuities, etc, the implicit relations may not require such tools or concepts. Indeed, if one rewrites (2.6) into the equivalent form (2.4) with **continuous** function $\mathbf{G}$, then one can avoid the use of such tools and simply stay at the level of continuous functions.

Let us now state the assumptions on the form of implicit law described by $\mathbf{G}$ more precisely. Heuristically, we want the model to satisfy the following principles: zero stress and zero shear must be an admissible configuration, the shear is nondecreasing with respect to the stress[5] and finally, the quantity (energy dissipation) $\xi = \mathsf{S} \cdot \mathsf{D}$ must provide compatible information about the stress and the shear. To formulate it mathematically, it seems to be quite natural to employ the notion of maximal monotone graphs. Hence, defining the graph $\mathcal{A}$ by

$$(\mathsf{D}, \mathsf{S}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathsf{G}(\mathsf{D}, \mathsf{S}) = \mathbf{0}\,, \tag{2.9}$$

we put the following assumptions on $\mathcal{A}$:

(i) $\mathcal{A}$ *comes through the origin:* $(\mathbf{0}, \mathbf{0}) \in \mathcal{A}$.

(ii) $\mathcal{A}$ *is a monotone graph:*

$$(\mathsf{S}_1 - \mathsf{S}_2) \cdot (\mathsf{D}_1 - \mathsf{D}_2) \geq 0 \qquad \text{for all } (\mathsf{D}_1, \mathsf{S}_1), (\mathsf{D}_2, \mathsf{S}_2) \in \mathcal{A}.$$

(iii) $\mathcal{A}$ *is a maximal monotone graph:* Let $(\mathsf{D}, \mathsf{S}) \in \mathbb{R}^{3\times 3}_{\text{sym}} \times \mathbb{R}^{3\times 3}_{\text{sym}}$ be given.

$$\text{If} \quad (\bar{\mathsf{S}} - \mathsf{S}) \cdot (\bar{\mathsf{D}} - \mathsf{D}) \geq 0 \quad \text{for all } (\bar{\mathsf{D}}, \bar{\mathsf{S}}) \in \mathcal{A} \quad \text{then } (\mathsf{D}, \mathsf{S}) \in \mathcal{A}.$$

(iv) $\mathcal{A}$ *is a ψ-graph:* There are non-negative constants m, $c_* > 0$ and a Young function ψ such that

$$\mathsf{S} \cdot \mathsf{D} \geq -m + c_*(\psi(|\mathsf{D}|) + \psi^*(|\mathsf{S}|)) \quad \text{for all } (\mathsf{D}, \mathsf{S}) \in \mathcal{A}\,.$$

Here, $\psi : \mathbb{R} \to \mathbb{R}_+$ is a Young function, i.e., ψ is an even continuous convex function such that

$$\lim_{s\to 0+} \frac{\psi(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{\psi(s)}{s} = \infty\,. \tag{2.10}$$

The convex conjugate function ψ^* is defined as the Legendre transform of ψ, i.e.,

$$\psi^*(s) := \sup_{\ell\in\mathbb{R}} \left(s \cdot \ell - \psi(\ell)\right)\,. \tag{2.11}$$

The assumptions (i), (ii) and (iv) really reflect the heuristical principles we require. The assumption (iii) (the maximality) is needed from the point of

[5] This is a reasonable assumption for fluids whose mechanical properties are not affected by the presence of some microstructure. Indeed, for example for flows of polymers, this principle can be wrong, since the stress is also affected by the configuration of macromolecules, see e.g. [9].

view of analysis but is also natural if one requires $\mathbf{G}$ to be continuous. In addition, for explicit models of the form (2.1), the condition (iii) automatically follows from (ii) provided that $\tilde{\mathbf{S}}$ is continuous, which can be obtained by using the monotone operator theory invented by Minty [69]. The importance of (iv) can be seen in (1.11), which implies

$$\sup_{t\in[0,T]} \int_\Omega |\boldsymbol{v}|^2 \, dx + \int_0^T \left(\int_\Omega \psi(|\mathbf{D}|) + \psi^*(|\mathbf{S}|) \, dx + \int_{\partial\Omega} |\boldsymbol{v}|^2 \, dS \right) dt < \infty \tag{2.12}$$

and gives the natural a priori estimates for $\mathbf{S}$ and $\boldsymbol{v}$.

We finish this section by showing the generality of the maximal monotone ψ graphs setting. First, for the power-law fluid given by (2.8) with $r > 1$, $\alpha_* = 0$ and $2\nu_* = 1$ for simplicity, we identify

$$\mathbf{S} = |\mathbf{D}|^{r-2}\mathbf{D} \quad \Longleftrightarrow \quad (\mathbf{S}, \mathbf{D}) \in \mathcal{A} \,. \tag{2.13}$$

Then one can observe that and $\mathcal{A}$ is maximal monotone ψ graph with ψ and ψ^* given as

$$\psi(s) := \frac{|s|^r}{r}, \qquad \psi^*(s) := \frac{|s|^{r'}}{r'}, \tag{2.14}$$

where $r' := r/(r-1)$ is the dual exponent to r. Furthermore, the presence of a general Young function in (iv) allows as to consider also the non-polynomial growth of the form

$$\mathbf{S} \sim (1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} \ln(1 + |\mathbf{D}|)\mathbf{D} \implies \psi(|\mathbf{D}|) \sim |\mathbf{D}|^r \ln(1 + |\mathbf{D}|) \,,$$

where $\sim$ denotes here the behaviour for large values of the shear rate $|\mathbf{D}|$. Even more, one can consider much wilder behaviour of the viscosity $\nu(|\mathbf{D}|)$. Indeed, taking for example a function $\nu(s)$ which satisfies with some $1 < r_1 < r_2 < \infty$

$$\liminf_{s\to\infty} \frac{\nu(s)}{s^{r_1-2}} \le 1, \qquad \limsup_{s\to\infty} \frac{\nu(s)}{s^{r_2-2}} \ge 1,$$

we get for the model (2.7) that $\mathbf{S}\cdot\mathbf{D}$ oscillates between $|\mathbf{D}|^{r_1}$ and $|\mathbf{D}|^{r_2}$.

Finally, to finish this part we show that for reasonable function ν and arbitrary $\tau_* \ge 0$, the graph $\mathcal{A}$ given implicitly by (2.5) is a maximal monotone ψ with a proper ψ. Notice that for $\tau_* = 0$ the graph $\mathcal{A}$ reduces to the explicit model (2.7).

Lemma 2.1. *Let $\tau_* \geq 0$, $\nu \in \mathcal{C}(0,\infty)$ be nonnegative and the function $g(s) := 2\nu(s)s$ be nondecreasing and fulfil*

$$\lim_{s\to 0_+} g(s) = 0, \qquad \lim_{s\to\infty} g(s) = \infty. \tag{2.15}$$

Then the graph $\mathcal{A}$ given by (2.5) *is maximal monotone ψ-graph with ψ given as*

$$\psi(t) := \int_0^{|t|} 2g(s)\, ds.$$

Proof. We follow the proof of [**B1**, Lemma 1.1], where we however considered only ν's having polynomial growth. The property (i) is obviously valid. Next, we focus of the monotonicity, i.e., the property (ii). Let $(\mathbf{S}_1, \mathbf{D}_1), (\mathbf{S}_2, \mathbf{D}_2) \in \mathcal{A}$. Then, if $|\mathbf{S}_1| \leq \tau_*$ and $|\mathbf{S}_2| \leq \tau_*$ then $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{0}$ and $(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) = 0$. In case that $|\mathbf{S}_2| \leq \tau_* < |\mathbf{S}_2|$, then $\mathbf{D}_1 = \mathbf{0}$ and we can use (2.6) to deduce

$$(\mathbf{S}_2 - \mathbf{S}_1)\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \mathbf{S}_2 \cdot \mathbf{D}_2 - \mathbf{S}_1 \cdot \mathbf{D}_2 = \tau_*|\mathbf{D}_2| + 2\nu(|\mathbf{D}|)|\mathbf{D}|^2 + \mathbf{S}_1 \cdot \mathbf{D}_2 \geq 0,$$

where the last inequality follows from the fact that $|\mathbf{S}_1| \leq \tau_*$ and nonnegativity of ν. Finally, if $|\mathbf{S}_1|, |\mathbf{S}_2| > \tau_*$ we can use (2.6) and recall the definition of g to obtain

$$\begin{aligned}
&(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2)\\
&= \tau_* \left(\frac{\mathbf{D}_1}{|\mathbf{D}_1|} - \frac{\mathbf{D}_2}{|\mathbf{D}_2|}\right)\cdot(\mathbf{D}_1 - \mathbf{D}_2) + 2\left(\nu(|\mathbf{D}_1|)\mathbf{D}_1 - \nu(|\mathbf{D}_2|)\mathbf{D}_2\right)\cdot(\mathbf{D}_1 - \mathbf{D}_2)\\
&= \tau^* \left(|\mathbf{D}_1| + |\mathbf{D}_2|\right) + g(|\mathbf{D}_1|)|\mathbf{D}_1| + g(|\mathbf{D}_2|)|\mathbf{D}_2|\\
&\qquad - \mathbf{D}_1 \cdot \mathbf{D}_2 \left(\frac{\tau_*}{|\mathbf{D}_1|} + \frac{\tau_*}{|\mathbf{D}_2|} + 2\nu(|\mathbf{D}_1|) + 2\nu(|\mathbf{D}_2|)\right)\\
&\geq \tau^* \left(|\mathbf{D}_1| + |\mathbf{D}_2|\right) + g(|\mathbf{D}_1|)|\mathbf{D}_1| + g(|\mathbf{D}_2|)|\mathbf{D}_2|\\
&\qquad - |\mathbf{D}_1||\mathbf{D}_2| \left(\frac{\tau_*}{|\mathbf{D}_1|} + \frac{\tau_*}{|\mathbf{D}_2|} + 2\nu(|\mathbf{D}_1|) + 2\nu(|\mathbf{D}_2|)\right)\\
&= g(|\mathbf{D}_1|)|\mathbf{D}_1| + g(|\mathbf{D}_2|)|\mathbf{D}_2| - g(|\mathbf{D}_1|)|\mathbf{D}_2| - g(|\mathbf{D}_2|)|\mathbf{D}_1|\\
&= (g(|\mathbf{D}_1|) - g(|\mathbf{D}_2|))(|\mathbf{D}_1| - |\mathbf{D}_2|) \geq 0,
\end{aligned}$$

where the last inequality follows from the fact that g is nondecreasing. Thus, the graph is monotone.

To show the maximal monotonicity, i.e., the property (iii), we rewrite (2.4) into the equivalent form

$$2\nu(|\mathsf{D}|)\mathsf{D} = \frac{(|\mathsf{S}| - \tau_*)_+}{|\mathsf{S}|}\mathsf{S} \tag{2.16}$$

Notice that the function on the right hand side is well defined for all $\mathsf{S} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. In addition, using the assumption (2.15), the continuity of ν and the fixed point theorem in finite dimension, it is standard to show that for arbitrary $\mathsf{S} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ there exists $\mathsf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ fulfilling (2.16). Hence, let us consider the couple (S, D) fulfilling assumptions of (iii). Then, we define $\mathsf{S}^n := \mathsf{S} - n^{-1}\mathsf{W}$ with arbitrary $\mathsf{W} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. Then using the argument above, we can find D^n fulfilling

$$2\nu(|\mathsf{D}^n|)\mathsf{D}^n = \frac{(|\mathsf{S}^n| - \tau_*)_+}{|\mathsf{S}^n|}\mathsf{S}^n\,, \tag{2.17}$$

which means that $(\mathsf{S}^n, \mathsf{D}^n) \in \mathcal{A}$. In addition, since D^n is a bounded sequence in $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ (it follows from (2.15)), then for a subsequence we have $\mathsf{D}^n \to \overline{\mathsf{D}}$. Consequently using the continuity of ν, (2.15) and the fact that $\mathsf{S}^n \to \mathsf{S}$, we obtain from (2.17) that

$$2\nu(|\overline{\mathsf{D}}|)\overline{\mathsf{D}} = \frac{(|\mathsf{S}| - \tau_*)_+}{|\mathsf{S}|}\mathsf{S} \tag{2.18}$$

and consequently also $(\mathsf{S}, \overline{\mathsf{D}}) \in \mathcal{A}$. Finally, using the assumption on (S, D) we have

$$0 \le n(\mathsf{S} - \mathsf{S}^n) \cdot (\mathsf{D} - \mathsf{D}^n) = \mathsf{W} \cdot (\mathsf{D} - \mathsf{D}^n) \overset{n\to\infty}{\to} \mathsf{W} \cdot (\mathsf{D} - \overline{\mathsf{D}}).$$

Since W is arbitrary, we have $\mathsf{D} = \overline{\mathsf{D}}$ and consequently $(\mathsf{S}, \mathsf{D}) \in \mathcal{A}$.

Finally, we focus on (iv). First, it follows from monotonicity of g and the assumption (2.15) that ψ is a Young function. Moreover, for any s we have the following relation (this is a consequence of the definition of ψ^*)

$$\psi(s) + \psi^*(2g(s)) = 2g(s)s. \tag{2.19}$$

Then for arbitrary $(\mathsf{S}, \mathsf{D}) \in \mathcal{A}$, we have that

$$\mathsf{S} \cdot \mathsf{D} = \tau_*|\mathsf{D}| + g(|\mathsf{D}|)|\mathsf{D}| \ge \frac{1}{2}\left(\psi(|\mathsf{D}|) + \psi^*(2g(|\mathsf{D}|))\right), \tag{2.20}$$

where for the second inequality we used (2.19). Next, if $|\mathsf{S}| \le 2\tau_*$, we have (using the fact that ψ^* is nondecreasing)

$$\mathsf{S} \cdot \mathsf{D} \ge 0 = \psi^*(|\mathsf{S}|) - \psi^*(|\mathsf{S}|) \ge \psi^*(|\mathsf{S}|) - \psi^*(2\tau_*). \tag{2.21}$$

Finally, if $|\mathsf{S}| > 2\tau^*$, then it follows from (2.6) that

$$2\tau_* \le |\mathsf{S}| = \tau^* + g(|\mathsf{D}|) \qquad \Longrightarrow \qquad |\mathsf{S}| \le 2g(|\mathsf{D}|).$$

Therefore, we have

$$\psi^*(|\mathsf{S}|) \le \psi^*(2g(|\mathsf{D}|)),$$

which combined with (2.21) and (2.20) leads to

$$\mathsf{S} \cdot \mathsf{D} \ge -\psi^*(2\tau_*) + \frac{1}{2}\left(\psi(|\mathsf{D}|) + \psi^*(|\mathsf{S}|)\right),$$

which is (iv). □

2.2 Notion of solution

Here, we give the precise meaning of the notion of weak solution to (1.1) completed by the Navier boundary conditions (1.6) and (1.8), the initial condition (1.10) and the implicit constitutive law (1.12), which impose the maximal monotone ψ graph. To do so, we recall some necessary mathematical tools and notions. For the sake of generality, we switch here to the general d-dimensional setting but still have in mind the most interesting three-dimensional case. Thus, we shall assume that $\Omega \subset \mathbb{R}^d$, $d > 1$, is a bounded domain with $\mathcal{C}^{1,1}$-boundary $\partial\Omega$, we denote $Q := (0,T) \times \Omega$ and $\Gamma := (0,T) \times \partial\Omega$. For $q \in [1,\infty]$ we define the Lebesgue spaces $L^q(\Omega)$ and the Sobolev spaces $W^{1,q}(\Omega)$ in a standard way. Further, if X, Y are Banach spaces, then $X^d := X \times \cdots \times X$ and we use X^* for dual space to X. In addition, we usually write $\langle f, g\rangle$ instead of $\langle f, g\rangle_{(X^*,X)}$ whenever there is no confusion what is the underlying space in the duality pairing. Further, $L^q(0,T;Y)$ denotes the standard Bochner space and $\mathcal{C}_{\mathrm{weak}}(0,T;L^q(\Omega))$ consists all $u \in L^\infty(0,T;L^q(\Omega))$ for which the function $t \mapsto \int_\Omega u(t)\varphi\, dx$ is continuous in $[0,T]$ for all $\varphi \in \mathcal{C}(\overline{\Omega})$.

Next, since we deal with the Navier boundary conditions, we introduce the subspaces of vector-valued Sobolev functions which have zero normal component on the boundary. First, we define in a standard way for any $p \in [1,\infty)$

$$L^q_{\boldsymbol{n},\mathrm{div}} := \overline{\{\boldsymbol{v} \in \mathcal{D}(\Omega)^d;\ \mathrm{div}\,\boldsymbol{v} = 0\}}^{\|\cdot\|_q}.$$

Then by $\mathcal{V}$ and $\mathcal{V}_{\mathrm{div}}$ we denote

$$\mathcal{V} := \{\boldsymbol{v} \in W^{d+2,2}(\Omega)^d;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \quad \mathcal{V}_{\mathrm{div}} := \mathcal{V} \cap L^2_{\boldsymbol{n},\mathrm{div}}.$$

Note, that $\mathcal{V} \subset W^{1,\infty}(\Omega)^d$ and therefore we can finally for any $q \in [1,\infty)$ introduce the following spaces

$$\begin{aligned} W^{1,q}_{\boldsymbol{n}} &:= \overline{\mathcal{V}}^{\|\cdot\|_{1,q}},\ W^{-1,q'}_{\boldsymbol{n}} := \left(W^{1,q}_{\boldsymbol{n}}\right)^* \quad (q' = q/(q-1)) \\ W^{1,q}_{\boldsymbol{n},\mathrm{div}} &:= \overline{\mathcal{V}_{\mathrm{div}}}^{\|\cdot\|_{1,q}}, W^{-1,q'}_{\boldsymbol{n},\mathrm{div}} := \left(W^{1,q}_{\boldsymbol{n},\mathrm{div}}\right)^*. \end{aligned}$$

Due to the fact that we deal with a possibly non-polynomial dependence, which is described by a general function ψ, we shall also require some additional assumptions on it. Hence, we say that a Young function ψ satisfies Δ_2-condition if there exist $C_1 > 0$ and $C_2 > 0$ such that for all $s \in \mathbb{R}$ we have

$$\psi(2s) \leq C_1\psi(s) + C_2. \tag{2.22}$$

The Δ_2-condition plays the crucial role in the properties of Orlicz spaces induced by ψ, which we however do not discuss here. We just recall the important fact (see [80, Chapter II, Cor. 5]) that if ψ and ψ^* satisfy Δ_2-condition then it follows that for certain $1 < q \leq r < \infty$ and positive constants c_1, c_1^*, c_2, c_2^*, c_3, c_3^*, c_4 and c_4^*

$$\begin{aligned} c_1 s^q - c_2 &\leq \psi(s) \leq c_3 s^r + c_4, \\ c_1^* s^{r'} - c_2^* &\leq \psi^*(s) \leq c_3^* s^{q'} + c_4^*, \end{aligned} \tag{2.23}$$

We reserve in what follows the notation for powers q and r through (2.23). Note that if $\psi(t) \sim t^r$ then $q = r$ in (2.23). Thus, we can define the notion of a weak solution in a relatively standard way as follows.

Definition 2.1. *Assume Ω is a Lipschitz set, $\mathcal{A}$ is a maximal monotone ψ graph with ψ satisfying* (2.23) *and*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}, \quad \mathbf{b} \in L^{q'}(0,T;W^{-1,q'}_{\boldsymbol{n}}) \ \text{ and } \ \gamma_* \geq 0\,, \tag{2.24}$$

We say that a triple $(p, \boldsymbol{v}, \mathbf{S})$ is a weak solution to (1.1), (1.6), (1.8), (1.10) *and* (1.12) *if*

$$\boldsymbol{v} \in \mathcal{C}_{weak}(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^q(0,T;W^{1,q}_{\mathbf{n},\mathrm{div}}),\ \mathbf{S} \in L^{r'}(Q), \tag{2.25}$$

$$\partial_t \boldsymbol{v} \in L^1(0,T;\mathcal{V}^*),\ p \in L^1(Q), \tag{2.26}$$

$$\int_Q \psi(|\mathbf{D}(\boldsymbol{v})|) + \psi^*(|\mathbf{S}|)\, dx\, dt + \gamma_* \int_\Gamma |\boldsymbol{v}|^2\, dS\, dt < \infty, \tag{2.27}$$

the initial condition is attained in the following sense

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2 = 0, \tag{2.28}$$

the equation (1.1) *is satisfied in the following sense: For all* $\boldsymbol{w}\in\mathcal{V}$ *and almost all* $t\in(0,T)$ *there holds*

$$\begin{aligned}\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle + \int_\Omega (\mathbf{S}-(\boldsymbol{v}\otimes\boldsymbol{v}))\cdot \mathbf{D}(\boldsymbol{w})\,dx + \gamma_* \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{w}\,dS \\ = \langle \mathbf{b},\boldsymbol{w}\rangle + \int_\Omega p \operatorname{div}\boldsymbol{w}\,dx\end{aligned} \tag{2.29}$$

and (1.12) *is satisfied almost everywhere in* Q*, i.e., for almost all* $(t,x)\in Q$ *there holds*

$$(\mathbf{D}(\boldsymbol{v}(t,x)),\mathbf{S}(t,x))\in\mathcal{A}. \tag{2.30}$$

In addition, we say that the weak solution satisfies the energy inequality if for all $\tau\in(0,T)$ *there holds*

$$\begin{aligned}\|\boldsymbol{v}(\tau)\|_2^2 + 2\int_0^\tau \left(\int_\Omega \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v})\,dx + \gamma_*\int_{\partial\Omega} |\boldsymbol{v}|^2\,dS\right)\,dt \\ \le \|\boldsymbol{v}_0\|_2^2 + 2\int_0^\tau \langle \mathbf{b},\boldsymbol{v}\rangle\,dt.\end{aligned} \tag{2.31}$$

The definition above seems to be an optimal setting if one deals with the concept of a weak solution. Frequently, the pressure p is omitted in the weak formulation by considering $\boldsymbol{w}\in\mathcal{V}_{\mathrm{div}}$ in (2.29). The reason is twofold. First, it is an simplification since we have not so many unknowns and the pressure then can be “reconstructed" at least as a distribution. Second, in some cases (mainly for Dirichlet boundary conditions) we do not know whether the pressure exists as an integrable function. But since we deal here only with the Navier boundary conditions for which we are able to get the existence of the pressure, we directly include it into the definition of a weak solution.

An “additional" information (2.31), which is automatically fulfilled with the equality sign whenever the solution is regular enough, may play the role of an selector among all weak solutions. Furthermore, there is a stronger concept of “selector" called a **suitable weak solution**, which is a solution that satisfies in sense of distributions

$$\partial_t|\boldsymbol{v}|^2 + \operatorname{div}\left(\boldsymbol{v}(|\boldsymbol{v}|^2+2p)\right) - 2\left(\operatorname{div}(\mathbf{S}\boldsymbol{v}) - \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}) + \mathbf{b}\right) \le 0 \qquad \text{in Q.} \tag{2.32}$$

The above inequality can be formally obtained by taking the scalar product of (1.1) with $\boldsymbol{v}$ and is essential for proving partial regularity results. However, while the notion of weak solution is admissible for any choice of ψ, the inequality (2.32) is well defined under the assumption that $(|\boldsymbol{v}|(|\boldsymbol{v}|^2 + |p|))$ is an integrable function, which requires certain restriction on the value of q in (2.23).

2.3 Description of results obtained in [B1, B2, B4]

First, we give a short survey about the available results answering the question about the existence and the uniqueness of a solution. The mathematical theory for (1.1) started already in 1934 when Leray in his work [52] considered the Cauchy problem for Navier–Stokes equations, i.e., $\Omega = \mathbb{R}^d$ and $\mathsf{S} = 2\nu_* \mathsf{D}(\boldsymbol{v})$ and proved the existence of classical (smooth) solution in dimension $d = 2$ and the existence of a **turbulent** solution for dimension three. His concept of turbulent solution is nowadays called weak solution. This result was later extended to bounded domains subjected by Dirichlet conditions by Hopf in [41]. The existence theory is based on the proper approximative scheme and the only difficulty arises due to the presence of the nonlinear term of the lower order, i.e., the convective term $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$. Nevertheless, due to the Aubin–Lions compactness argument, one can easily treat this term and get the existence of a solution in this case.

However, concerning the regularity of the solution in the three dimensional setting the problem remains still open. Besides the conditional regularity results[6] there are partial regularity result [32, 22]. In [32] there is shown that every weak solution[7] satisfying (2.31) is smooth in $(I \times \Omega)$, where the Hausdorff dimension of $(0, T) \setminus I$ is less than $1/2$ and in [22] it is shown that the suitable weak solution, i.e., the solution satisfying (2.32) is smooth in Q except the set, which has the Hausdorff dimension less than one.

A completely different story is the regularity in two dimensional setting due to the special structure of the convective term and the full regularity of the solution as well as its uniqueness can be obtained. In addition, if one neglects

[6]These are the results that guarantee smooth solution under some additional hypothetical qualitative property imposed on the solution. Since we are not interested in this class of results we do not provide any reference here.

[7]It is proven for Dirichlet boundary condition but holds as well for the Navier boundary condition.

the convective term then in arbitrary dimension we have a sharp regularity theory due to [83] (see also [84] for more general result).

2.3.1 Existence results for nonlinear models

First systematic study of the nonlinear models started by the pioneering works of Ladyzhenskaya [46, 47, 48] who considered the nonlinear explicit models (2.7) and focused mainly on the power-law like models (2.8). In the model of type (2.8) one has to deal with two nonlinearities, the convective term and the nonlinearity in the constitutive equation. From the mathematical point of view, it can be split into two cases. First, when the convective term can be understood as a "compact perturbation" called subcritical (or critical) and the second, when the convective term behaves worse than the nonlinearity in $\mathbf{S}$. Mathematically speaking, the first case means that we can test (1.1) by the solution, i.e., to set $\boldsymbol{w} := \boldsymbol{v}$ in (2.29) and integrate with respect to time. And the second case, when $\boldsymbol{v}$ is not sufficiently regular to be used as a test function in (2.29). This restriction just means that

$$(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{v} \in L^1(Q).$$

Thanks to a priori estimate (2.12) and the polynomial r-growth of the models considered by Ladyzhenskaya, it then reduces to cases:

$$r > \frac{3d+2}{d+2} \qquad \text{subcritical,} \tag{2.33}$$

$$r = \frac{3d+2}{d+2} \qquad \text{critical,} \tag{2.34}$$

$$r \in \left(\frac{2d}{d+2}, \frac{3d+2}{d+2}\right) \qquad \text{supercritical.} \tag{2.35}$$

The lower bound in (2.35) is caused by the fact that we need at least that $\boldsymbol{v}$ is compact in $L^2(Q)$ to be able to handle the convective term. Note that for lower values of r's there is no available[8] theory.

For the existence of solution, Ladyzhenskaya applied the Minty method [69] to the equation (1.1) to identify the nonlinearity $\mathbf{S}$ to get the existence of a solution. This method however requires that if $\{\boldsymbol{v}^n\}_{n=1}^{\infty}$ is a sequence of

[8] Global in time or large data results.

approximative solutions, $\boldsymbol{v}$ is its weak limit and we consider the law $\mathsf{S}^n = \nu(|\mathsf{D}(\boldsymbol{v}^n)|)\mathsf{D}(\boldsymbol{v}^n)$ then

$$\limsup_{n\to\infty} \int_Q \mathsf{S}^n \cdot \mathsf{D}(\boldsymbol{v}^n - \boldsymbol{v})\, dx\, dt \le 0 \tag{2.36}$$

The relation (2.36) is usually obtained by testing by solution, which is however possible only in cases (2.33) and (2.34). In these cases for power-law like models and Dirichlet boundary conditions, Ladyzhenskaya obtained the existence of a weak solution. Since she was not interested in the Navier boundary conditions, they are not covered in her works but the method of the proof would be the same. In addition, if one considers the generalized Stokes problem, i.e., the problem (1.1) without the convective term $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$, this method can be applied for all $r \in (1, \infty)$ as well.

It is remarkable that the existence result of Ladyzhenskaya does not cover the case $r = 2$ if dimension is higher than two, which corresponds to the Navier–Stokes equations. This gap was was removed in [60], where for the spatially periodic setting the authors established the existence of a weak solution for $r > 3d/(d + 2)$. They introduced the higher differentiability method and showed that for such r's, the solution even belongs to $L^1(0, T; W^{1+\varepsilon,1}(\Omega))$, which then leads to the compactness of $\nabla \boldsymbol{v}^n$ and to identification of the nonlinearity since the model is explicit. The bound imposed on r is identical to the same bound when one wants to speak about suitable weak solutions fulfilling (2.32) and which is that $\boldsymbol{v} \in L^3(Q)^d$. The method in [60] however heavily used the periodic boundary conditions and it was extended to the Dirichlet boundary conditions in [61] but only for $r \ge 2$ and dimension $d = 3$. The method in [60] does not use the Minty method, however it needs a kind of uniform monotonicity to obtain the higher differentiability estimates.

Later on, in [35] the L^∞-truncation method[9] was adapted to fluid setting and the authors established the existence of a weak solution for power-like models with $r > (2d + 2)/(d + 2)$ and perfect slip boundary conditions. This result can be further directly extended also to the case of the Navier slip boundary conditions. The method is based on using an L^∞ approximation of $\boldsymbol{v}$ in (2.29), which is possible as far as the convective term is integrable, i.e., it requires

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \in L^1(Q).$$

[9]This method was first used in the context of elliptic and parabolic equations with bad (L^1) data [11, 10].

Since, $|\nabla \boldsymbol{v}| \in L^r(Q)$, one needs that $|\boldsymbol{v}| \in L^{r'}(Q)$, which finally leads to the restriction $r > (2d+2)/(d+2)$. In addition, using such an approximation, one replaces (2.36) by

$$\limsup_{n\to\infty} \int_{Q\cap\{\varepsilon^{-1}\geq|\boldsymbol{v}^n-\boldsymbol{v}|>\varepsilon\}} \mathbf{S}^n \cdot \mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})\, dx\, dt \leq 0 \tag{2.37}$$

for arbitrary $\varepsilon > 0$. The property (2.37) is much weaker than (2.36) but still leads to the point-wise convergence of $\nabla \boldsymbol{v}^n$ provided that the operator is strictly monotone. The extension of this method also to the Dirichlet boundary condition was performed in [90].

The remaining gap, i.e., the case $r \in (2d/(d+2), (2d+2)/(d+2))$ was firstly treated in the steady case in [36], where the authors employed the Lipschitz approximation method[10] to get the existence for power-law like models with $r > 2d/(d+2)$, i.e., the most general result in the context of power-law like fluids (2.8). Note that the method again requires the strict monotonicity of the operator. This methods for steady flows was later extended in [27] also to the case when the power exponent depends on the spatial variable x. The most general result for unsteady flows with the power-law like rheology, i.e., (2.8) was finally proven by Diening, Růžička and Wolf [28], where the authors adapted the method of Lipschitz approximation (developed in [43] for parabolic equations) also to the case of incompressible fluids and established the existence result for power-law like models with the optimal range $r > 2d/(d+2)$ for Dirichlet boundary conditions (The very similar approach would also work for the Navier boundary condition.). The parabolic Lipschitz approximation method is based on setting $\boldsymbol{w} := \boldsymbol{v}_\lambda$ in (2.29), where $\boldsymbol{v}_\lambda$ is a proper Lipschitz (with respect to the spatial variable) approximation of $\boldsymbol{v}$. Such a setting then leads to

$$\limsup_{n\to\infty} \int_{Q\cap\{M(|\nabla\boldsymbol{v}^n|+|\nabla\boldsymbol{v}|)\leq\lambda\}} \mathbf{S}^n \cdot \mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})\, dx\, dt \leq 0, \tag{2.38}$$

where $\lambda > 0$ is arbitrary and M denotes the Hardy–Littlewood maximal function. Then from (2.38) and the strict monotonicity the authors deduced the point-wise convergence of $\nabla \boldsymbol{v}^n$. Nevertheless, the power-law like dependence with fixed r as well as the strict monotonicity and the continuity of the

[10]This method was developed by Acerbi and Fusco in [1] for studying the regularity of minimizers to certain elliptic problems.

operator reduces the applicability of the result only to the class of explicit models (2.8). For more references and more detail explanations for power-law like models we refer to the survey paper [64] or to [17, 15].

In the framework of implicitly constituted fluids, the most attention attracted the Bingham or the Herschel–Bulkley fluids, i.e., the models (2.5) $\nu(s) \sim s^{r-2}$ with $r = 2$ for Bingham fluids and $r \neq 2$ Herschel–Bulkley fluids. These fluids were analyzed in many works, see [29, 37, 47, 65, 81, 82], but under the assumption that the operator provides a kind of uniform monotonicity and still dealing with the r-growth. The existence results are merely based on the use of higher regularity of the solution and therefore all these results are subjected to the assumption $r > 3d/(d+2)$. The assumption on the uniform monotonicity was removed for steady case in [40] for $r > 3d/(d+2)$ and finally in steady case, the authors in [15] established the existence theory for steady models (2.5) with $\nu(s) \sim s^{r-2}$ for all $r > 2d/(d+2)$, so the most optimal setting but still requiring the strict monotonicity.

The result presented in [**B1**] goes much beyond all of the results presented above. First, it treats the general growth described by ψ, second, it does not require any kind of strict monotonicity but just the maximality of the monotone graph and last, it covers the full range of reasonable growth, i.e., for ψ satisfying (2.23) with $q > 2d/(d+2)$, which is just the condition to get the compactness of the convective term. More precisely, the result of [**B1**] is:

Theorem 2.1. *Let $\mathcal{A}$ be a maximal monotone ψ graph with ψ and ψ^* satisfying Δ_2-condition and fulfilling* (2.23) *with*

$$q > \frac{2d}{d+2}. \tag{2.39}$$

Then for any $\Omega \in \mathcal{C}^{1,1}$ and $T \in (0,\infty)$ and for arbitrary $\boldsymbol{v}_0$, $\mathbf{b}$ and γ_* satisfying (2.24) *there exists a weak solution in the sense of Definition 2.1 fulfilling the energy inequality* (2.31). *In addition, if*

$$q > \frac{3d}{d+2} \tag{2.40}$$

then there exists a suitable weak solution, i.e., it satisfies the inequality (2.32).

To end this subsection, we just give few remarks about the main result. The restriction (2.39) on the parameter q is due to required compact embedding into $L^2(0,T;L^2(\Omega)^d)$ used in the identification of the limit in the quadratic term. If we consider Stokes like systems we can relax the assumption on q and require that $q \geq 1$.

Next comment concerns the role of the boundary condition. We consider the Navier slip boundary conditions (1.8) for several reasons. First of all, we are able to construct the pressure p as an integrable function (while p in [28] and other studies analyzing time-dependent three-dimensional flows of an incompressible non-Newtonian fluid subject to the no-slip boundary condition is merely a distribution, with respect to the time variable, see [90]). Navier's slip boundary condition (1.8) thus helps us to avoid the splitting of the pressure (performed in [28, 90]) into the regular part and the distribution, which brings additional technical difficulties that we did not want to mix up with the other tools developed here. It is also worth observing that the analysis can be developed for boundary conditions different from (1.9), which has also an implicit character and allows the so-called threshold slip boundary conditions, see for example [19] and also the extension in [66]. Furthermore, Theorem 2.1 can be also extended to the case of Dirichlet boundary condition by a proper decomposition of the pressure as invented in [90]. It is also necessary to recognize that in order to obtain integrable pressure, one has to require $\mathcal{C}^{1,1}$-regularity of the boundary and the corresponding theory for the Neumann problem for the Laplace operator in Orlicz space setting when introducing the pressure. Nevertheless, the result of Theorem 2.1 remains valid also for Lipschitz domains with the only modification that we do not have pressure as integrable functions and we require $\boldsymbol{w} \in \mathcal{V}_{\mathrm{div}}$ in (2.29). Such a result can be obtained by methods developed in [**B1**] and [13].

In addition, as a "by-product" of the main theorem in [**B1**], we obtained the following key essential results needed for the analysis and which are of independent interest and are used now in subsequent papers. The first is a generalization of the Minty method also to the setting of maximal monotone graph, which was not known in the generality needed in [**B1**].

Lemma 2.1. *Let $\mathcal{A}$ be maximal monotone ψ-graph with ψ and ψ^* satisfying Δ_2-condition and assume that there are sequences $\{\mathsf{S}^n\}_{n=1}^{\infty}$ and $\{\mathsf{D}^n\}_{n=1}^{\infty}$ such that $(\mathsf{D}^n(t,x), \mathsf{S}^n(t,x)) \in \mathcal{A}$ for a.a. $(t,x) \in Q$, we have the weak convergence* *results* $\mathbf{D}^n \rightharpoonup \mathbf{D}$ *and* $\mathbf{S}^n \rightharpoonup \mathbf{S}$ *weakly in* $L^1(Q)^{d\times d}$ *and in addition satisfy*

$$\sup_n \int_Q \psi^*(|\mathbf{S}^n|) + \psi(|\mathbf{D}^n|)\, dx\, dt < \infty,$$

$$\limsup_{n\to\infty} \int_Q \mathbf{S}^n \cdot \mathbf{D}^n\, dx\, dt \le \int_Q \mathbf{S} \cdot \mathbf{D}\, dx\, dt.$$

Then for almost all $(t,x) \in Q$ *we have*

$$(\mathbf{D}(t,x), \mathbf{S}(t,x)) \in \mathcal{A}.$$

Although this lemma is not stated explicitly in [**B1**], it is used there implicitly. We would like to point out, that it works with general functions ψ and we do not need any kind of strict monotonicity.

Second result is the extension of the Lipschitz approximation method from the polynomial growth to a general Young function ψ.

Lemma 2.2. *Let* $\Omega \subset \mathbb{R}^d$ *be an open bounded set and* $T > 0$ *be the length of the time interval. Assume that* ψ *and* ψ^* *are Young functions satisfying* Δ_2 *condition. Then, for any functions* $\mathbf{H}, \bar{\mathbf{H}}$ *and arbitrary sequences* $\{\boldsymbol{u}^n\}_{n=1}^\infty$ *and* $\{\mathbf{H}^n\}_{n=1}^\infty$ *we set*

$$a^n := |\mathbf{H}^n| + |\mathbf{H}| + |\bar{\mathbf{H}}| \quad \text{and} \quad b^n := |\mathbf{D}(\boldsymbol{u}^n)|$$

such that for certain $C^* > 1$

$$\begin{aligned} &\int_Q \psi^*(a^n) + \psi(b^n)\, dx\, dt + \sup_{t\in(0,T)} \|\boldsymbol{u}^n(t)\|_2^2 \le C^*, \\ &\boldsymbol{u}^n \to \mathbf{0} \qquad a.e.\ in\ Q := (0,T)\times\Omega. \end{aligned} \tag{2.41}$$

In addition, let $\{\mathbf{G}^n\}_{n=1}^\infty$ *and* $\{\boldsymbol{f}^n\}_{n=1}^\infty$ *be such that* $\mathbf{G}^n$ *is symmetric and*

$$\mathbf{G}^n \to \mathbf{0} \qquad \text{strongly in } L^1(Q)^{d\times d}, \tag{2.42}$$

$$\boldsymbol{f}^n \to \mathbf{0} \qquad \text{strongly in } L^1(Q)^{d}, \tag{2.43}$$

and that the following identity holds in $\mathcal{D}'(Q)^d$

$$\partial_t \boldsymbol{u}^n + \operatorname{div}(\mathbf{H}^n - \mathbf{H} + \mathbf{G}^n) = \boldsymbol{f}^n. \tag{2.44}$$

Then there exists $\beta > 0$ such that for arbitrary open $Q_h \subset\subset Q$ and for arbitrary $\lambda^* \in (\lambda_{min}, \infty)$ with λ_{min} such that $\psi(\lambda_{min}) = \lambda_{min}$ and arbitrary $k \in \mathbb{N}$ there exists a sequence of $\{\lambda_k^n\}_{n=1}^\infty$ and the sequence of open sets $\{E_k^n\}_{n=1}^\infty$, $E_k^n \subset Q$ and a sequence $\{\boldsymbol{u}^{n,k}\}_{n=1}^\infty$ bounded in $L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ such that for any $1 \le s < \infty$

$$\lambda_k^n \in [\lambda^*, (c_3 + c_4/\lambda^r_{min})^{\frac{r^k-1}{r-1}} (\lambda^*)^{r^k}], \tag{2.45}$$

$$\|\mathbf{D}(\boldsymbol{u}^{n,k})\|_{L^\infty(Q_h)} \le C(h,\Omega)\lambda_k^n, \tag{2.46}$$

$$\limsup_{n\to\infty} |Q_h \cap E_k^n| \le C(h,\Omega)\frac{C^*}{\psi(\lambda^*)} \tag{2.47}$$

and

$$\boldsymbol{u}^{n,k} = \boldsymbol{u}^n \qquad \text{in } Q_h \setminus E_k^n, \tag{2.48}$$

$$\boldsymbol{u}^{n,k} \to \mathbf{0} \qquad \text{strongly in } L^s(Q_h)^d. \tag{2.49}$$

Moreover, for all $g \in \mathcal{D}(Q_h)$ the following estimates hold

$$\begin{aligned}\limsup_{n\to\infty} \int_{Q_h\cap E_k^n} \left(|\mathbf{H}^n| + |\mathbf{H}| + |\bar{\mathbf{H}}|\right) |\mathbf{D}(\boldsymbol{u}^{n,k})| \, dx \, dt \\ \le C(h, C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)} + \frac{1}{k^\beta}\right),\end{aligned} \tag{2.50}$$

$$-\liminf_{n\to\infty} \int_0^T \langle \partial_t \boldsymbol{u}^n, \boldsymbol{u}^{n,k} g\rangle \, dt \le C(g,h,C^*)\left(\frac{\lambda^*}{\psi(\lambda^*)} + \frac{1}{k}\right)^\beta. \tag{2.51}$$

This lemma is exactly used for obtaining (2.38) in our setting. Indeed, if $\boldsymbol{v}^n$ is an approximative solution, then $\boldsymbol{u}^n := \boldsymbol{v}^n - \boldsymbol{v}$, $\mathbf{G}^n := \boldsymbol{v}^n \otimes \boldsymbol{v}^n + p_1^n \mathbf{I} - \boldsymbol{v}\otimes\boldsymbol{v} - p_1\mathbf{I}$, $\mathbf{H}^n = \mathbf{S}^n + p_2^n\mathbf{I}$ and $\mathbf{H}$ is the corresponding weak limit and $\boldsymbol{f}^n$ can be some term coming from the approximation scheme. Here, we decomposed the pressure $p^n = p_1^n + p_2^n$ onto two parts, one, which has low integrability but is compact in L^1 - the part p_1^n, and the second part p_2^n which has the same integrability as $\mathbf{S}^n$. Then we can test the equation by $\boldsymbol{u}^{n,k}$, which has bounded gradient and using the properties (2.50) and (2.51) and letting $k \to \infty$ and $\lambda^* \to \infty$ we conclude that for any $\delta > 0$ there exists a measurable set $Q_\delta \subset Q$ such that $|Q \setminus Q_\delta| \le \delta$ and

$$\limsup_{n\to\infty} \int_{Q_\delta} \mathbf{S}^n \cdot \mathbf{D}^n \, dx \, dt \le \int_{Q_\delta} \mathbf{S}\cdot\mathbf{D} \, dx \, dt.$$

Hence, we can use Lemma 2.1 on the set Q_δ and let $\delta \to 0_+$ to obtain the result on the whole set Q.

Finally, in order to reconstruct the pressure and also to split the pressure onto two parts, we required the theory for Laplace equation with Neumann data, which was not known before.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^{1,1}$ domain. Assume that ψ is a Young function and that ψ and ψ^* satisfy Δ_2 condition. Then for arbitrary $f \in L^1(\Omega)$ with zero mean value there exists unique $u \in W^{2,1}(\Omega)$ fulfilling*

$$\begin{aligned} -\Delta u &= f && \text{a.e. in } \Omega, \\ \nabla u \cdot \boldsymbol{n} &= 0 && \text{a.e. on } \partial\Omega, \\ \int_\Omega \psi(|\nabla^2 u|)\,dx &\le C(\psi,\Omega)\left(\int_\Omega \psi(|f|)\,dx + 1\right) \end{aligned}$$

provided that the last integral is finite.

This lemma, or more precisely, the theory for $(-\Delta)^{-1}$ is the used to reconstruct the pressure, where the pressure is split such that it formally satisfies

$$\begin{aligned} -\Delta p_2 &= -\operatorname{div}\operatorname{div}\mathbf{S}, \\ -\Delta p_1 &= \operatorname{div}\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) \end{aligned}$$

Then using Lemma 2.3, one can deduce that (for $s \in (1,\infty)$ and ψ and ψ^* satisfying Δ_2-condition)

$$\begin{aligned} \int_Q \psi^*(|\mathbf{S}|) < \infty &\implies \int_Q \psi^*(|p_2|) < \infty, \\ \int_Q |\boldsymbol{v}|^{2s} < \infty &\implies \int_Q |p_1|^s < \infty. \end{aligned}$$

2.3.2 Uniqueness and regularity for nonlinear models

We already briefly discussed the regularity and the uniqueness issue for Navier–Stokes equations, i.e., the model (1.1) with $\mathbf{S} = 2\nu_*\mathbf{D}$. To summarize, we know that in two dimensional setting weak solution is unique and as smooth as data allows, while in three (and higher) dimensional setting both problems are widely open and the only available results are of the conditional type (small-data, special structure, hypothetical control of certain quantities, etc.). In model with a more general form of the Cauchy stress the situation is even worse. We have to face to two nonlinearities, the convective term and the viscous term. Nevertheless, in two dimensional setting, the convective term has a very special structure and therefore does not "contribute" to a possibly singular behaviour of the solution. In addition, although $\mathbf{S}$ can be of a nonlinear form, one can adapt the classical "two-dimensional" methods to obtain the smooth solution provided that the data are sufficiently regular, see e.g. [42, 90, 58]. Here, the data also means that $\mathbf{S}$ is sufficiently regular function of $\mathbf{D}$. Consequently, we also get the uniqueness of a weak solution fulfilling the energy inequality by standard methods. However, much more delicate is the situation when dimension is higher than two and we restrict ourselves from now to the most physical three-dimensional setting.

It is quite natural that if we require some additional qualitative properties for the weak solution, we need to impose more restriction on the constitutive law for $(\mathbf{S}, \mathbf{D})$, i.e., for the graph $\mathcal{A}$. To simplify the presentation, we consider from now that $\mathcal{A}$ is a maximal monotone r-graph, which means that it is a ψ graph with $\psi(t) := t^r$. Furthermore, we will assume that the graph satisfies in addition certain uniform monotonicity, i.e., we require that there exists a positive constant C_1 such that for all $(\mathbf{S}_1, \mathbf{D}_1), (\mathbf{S}_2, \mathbf{D}_2) \in \mathcal{A}$ we have

$$(\mathbf{S}_1 - \mathbf{S}_2) \cdot (\mathbf{D}_1 - \mathbf{D}_2) \geq C_1(1 + |\mathbf{D}_1| + |\mathbf{D}_2|)^{r-2}|\mathbf{D}_1 - \mathbf{D}_2|^2. \tag{2.52}$$

Notice that this kind of minimal assumption leads to the further regularity of $\boldsymbol{v}$ and the assumption (2.52) is still valid for implicitly constituted models (2.4) with $\nu(s) \sim (1 + s^2)^{(r-2)/2}$. On the other hand, if we require that also $\mathbf{S}$ has certain regularity properties then we also need to impose

$$|\mathbf{S}_1 - \mathbf{S}_2| \leq C_2(1 + |\mathbf{D}_1| + |\mathbf{D}_2|)^{r-2}|\mathbf{D}_1 - \mathbf{D}_2|, \tag{2.53}$$

valid again for all $(\mathbf{S}_i, \mathbf{D}_i) \in \mathcal{A}$. Hence, we see that $\mathbf{S}$ must be locally Lipschitz function of $\mathbf{D}$ and therefore cannot be satisfied by models of type (2.4) with $\tau_* > 0$.

2.3.3 Regularity results in three dimensions

The situation in three dimensional setting is quite similar as for the Navier–Stokes equations. Due to the r-growth and corresponding estimates, the convective term dictates the general structure of the problem and we can again split the problem to the subcritical case, which in dimension three corresponds[11] to $r > 11/5$, the critical case $r = 11/5$ and to the supercritical case, when $r < 11/5$. In the supercritical regime (which includes Navier–Stokes equations), the situation is the same as with Navier–Stokes equations. The problem of global regularity and uniqueness is an open problem and there are only small-data, short-time or other conditional results, which however we will not discuss here. We refer here only to an "optimal" result in [21], where the authors proved the partial regularity result and provided the estimate on size of singular times. Here, the word optimal means that it gives the same estimate, which is available for Navier–Stokes equation (see [32]), i.e., the case $r = 2$, and vanishes as $r \to 11/5_{-}$, which is expected behaviour at least for spatially periodic setting.

However, even in the subcritical case, or in the case of generalized Stokes problem, where we neglect the convective term, the situation concerning the existence of classical solution is totally different to the standard **linear** Stokes problem. Indeed, since we have a system of nonlinear equations with the elliptic operator not having the radial structure, we cannot use the classical result of Uhlenbeck [89] to obtain the existence of $\mathcal{C}^1$ solution which would then by standard methods lead to the existence of classical solution. In fact, this is one of the most challenging regularity open problems in the theory of fluid flow. Nevertheless, at least for the generalized Stokes system or for the full system in the subcritical regimes we can expect the same results as for general elliptic systems, i.e., the existence of a strong solution, which is the weak solution fulfilling in addition (for the models satisfying (2.52))

$$\int_Q (1 + |\mathbf{D}(\boldsymbol{v})|)^{r-2} |\nabla^2 \boldsymbol{v}|^2 \, dx \, dt < \infty \tag{2.54}$$

or in the steady case its steady counterpart.

Let us focus on the estimate (2.54). For the problem "without boundary", i.e., for the spatially periodic problem or the Cauchy problem, we can refer to the classical book [60] for the estimate (2.54) provided that the initial condition $\boldsymbol{v}_0 \in W^{1,2}(\Omega)^3$ and the right hand side $\mathbf{b} \in L^{\max\{2,p'\}}(Q)^3$. However, for the problems with physically reasonable boundary conditions, the estimate (2.54) is not know for $r \geq 11/5$. The reason is that when one localizes the equation

[11] This splitting can be done in general dimension and then it corresponds to the setting in (2.33)–(2.35).

with respect to the spatial variable, one has to deal also with the pressure. Note that for steady problem and $r \geq 9/5$ (which is the subcritical case for steady problems), we have local estimates (see [72]), i.e., for any $\Omega' \subset\subset \Omega$

$$\int_{\Omega'} (1 + |\mathbf{D}(\boldsymbol{v})|)^{r-2} |\nabla^2 \boldsymbol{v}|^2 \, dx \leq C(\Omega'). \tag{2.55}$$

For estimates up to the boundary, there are available many results in steady case but none of them seems to be optimal and compatible with the standard theory for elliptic systems, we refer e.g. to [57] and references therein.

However, in unsteady case, even the local (with respect to the spatial variable) estimate was not known for the whole range $r \geq 11/5$. The reason is that in unsteady setting, the pressure and the time derivative of the velocity field must be treated simultaneously during the localization, which causes an additional difficulties. The first result with the local estimates of the type (2.54) were proven in [61] for certain range of $r > r_0 > 11/5$. Moreover, these estimates in certain form were proven up to the boundary (the authors considered the Dirichlet boundary conditions, but the same procedure would work also for the Navier slip boundary conditions). This result was recently improved in [2], where the authors were able to shift the value of r to $r \geq 2.23 > 11/5$, i.e., almost the optimal range. Nevertheless, the estimates are only local in Ω. In addition, in order to estimate the pressure term, one certainly needs to impose the condition (2.53) and thus the implicit models are automatically excluded from the analysis[12]. The second possible attempt, how to obtain the regularity of the solution, is to improve the time regularity and then to use the elliptic regularity, which is locally in Ω available due to [72]. The advantage of this procedure is that one does not need to localize in Ω and one can just focus on the time evolution. The improved time regularity was already obtained by Ladyzhenskaya [49], where she got that for $r \geq 12/5$, initial data $\boldsymbol{v}_0 \in W^{1,p}(\Omega)^3$ and $\mathbf{b} \in L^2(Q)^3$, we have

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|\boldsymbol{v}(t)\|_{1,p} + \int_Q |\partial_t \boldsymbol{v}|^2 \, dx \, dt < \infty. \tag{2.56}$$

Consequently, having this estimate, we can use the steady theory to obtain also the spatial regularity results for all $r \geq 12/5$ without any need of difficult

[12] It is not the case when one deals with spatial periodic boundary conditions, see e.g. [65]

evolutionary pressure localization since from (2.56), we know that the time derivative is in fact an L^2 object. Furthermore, in case of non-smooth initial data, the above inequality can be localized and one can prove (for $r \geq 12/5$) the existence of a weak solution that fulfills for all $\tau > 0$

$$\operatorname*{ess\,sup}_{t\in(\tau,T)} \|\boldsymbol{v}(t)\|_{1,p} + \int_\tau^T \int_\Omega |\partial_t \boldsymbol{v}|^2 \, dx \, dt \leq C(\tau^{-1}). \tag{2.57}$$

2.3.4 Uniqueness results in three dimensions

The importance of regularity of the solution also immediately appears when one tries to obtain the uniqueness of the solution, provided that $r \geq 11/5$ (the natural (sub)-critical case) and (2.52), i.e., the natural requirements when dealing with the convective term. The starting point for the uniqueness of the solution is the following inequality, which is valid for all weak solutions. Here, $\boldsymbol{v}_i$ is an arbitrary weak solution corresponding to the right hand side $\mathbf{b}_i$.

$$\begin{aligned} &\frac{d}{dt}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_2^2 + \delta(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{1,2}^2 + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{1,r}^2) \\ &\qquad \leq C\left(\|\boldsymbol{v}_2\|_{1,p}^{\frac{2r}{2r-3}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_2^2 + \|\mathbf{b}_1 - \mathbf{b}_2\|_{W^{-1,r'}_{\boldsymbol{n},\mathrm{div}}}^{p'}\right). \end{aligned} \tag{2.58}$$

We refer to the proof of this inequality to [**B4**], where it is proven for very general boundary conditions, but was already obtained by Ladyzhenskaya for Dirichlet boundary conditions. Consequently, to obtain the uniqueness of solution, we can use the Gronwall lemma, provided that at least one weak solution satisfies

$$\boldsymbol{v} \in L^{\frac{2r}{2r-3}}(0,T;W^{1,r}(\Omega)^3). \tag{2.59}$$

Note that if $r \geq 5/2$ then (2.59) just follows from the definition of a weak solution. Furthermore, in case of initial data $\boldsymbol{v}_0 \in W^{1,r}(\Omega)^3$ and $r \geq 12/5$, one can use the result of Ladyzhenskaya (2.56) and to obtain (2.59). In case that $\boldsymbol{v}_0$ does not have enough regularity, we can still “localize" this procedure and for $r \geq 12/5$ obtain the uniqueness of solution in sense of trajectories, i.e., in case that two solutions $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ fulfills for some $t_1 < t_2$ that $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$ for all $t \in (t_1, t_2)$ then $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$ for all $t \geq t_2$. For this classical results, we refer to [49] and [50], or also to more recent results concerning the dynamics of (1.1) to [59, 62, 63], where the dynamics is exactly studied by using the uniqueness in sense of trajectories. However, any kind of estimates of type (2.56)–(2.57) or the **global** estimates of the form (2.54) (which would also guarantee (2.59)) were missing for the range $r \in [11/5, 12/5)$.

2.3.5 Presented results [B4, B2]

Here, we present our main novelties. We first focus on the uniqueness of the solution, which seems to be now completely resolved in the whole expected range of r's. The following results are quoted from [**B4**, subsection 3.3]. Although, [**B4**] deals only with explicit models, it requires just (2.52) and therefore can be easily adapted to our general setting. Furthermore, the results in [**B4**] are valid for much more general class of boundary conditions than the Navier slip boundary conditions. The improved integrability result reads as:

Theorem 2.2. *Let the maximal monotone r-graph $\mathcal{A}$ satisfy* (2.52), $r \geq 11/5$, $\mathbf{b} \in L^{r'}(0,T; W^{-1,r'}_{\boldsymbol{n},\mathrm{div}})$ *and* $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$. *In case that* $r < 5/2$ *assume in addition that* $\mathbf{b} \in N^{\delta,r'}(0,T; W^{-1,r'}_{\boldsymbol{n},\mathrm{div}})$ *with some*

$$\delta > (r-1)\left(\frac{5}{2r} - 1\right). \tag{2.60}$$

Then for an arbitrary weak solution $\boldsymbol{v}$ *to* (1.1) *the following holds true:*

1) *For any* $\tau \in (0,T)$ *we have that* $\boldsymbol{v} \in L^{\frac{2r}{2r-3}}(\tau,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}})$.

2) *If* $\boldsymbol{v}_0 \in W^{1,r}_{\boldsymbol{n},\mathrm{div}}$, *the conclusion holds for* $\tau = 0$ *as well.*

Here, the space $N^{\delta,r'}$ denotes the standard Nikolskii space[13] of fractional derivatives.

Theorem 2.2 has the important consequence. Recalling (2.59), we see that if the right hand side is sufficiently regular with respect to the time variable

[13]Nikolskii spaces are defined as follows. For any $p \in [1,\infty]$ and $\kappa \in (0,1]$, we say that $u \in N^{\kappa,p}(0,T;X)$ if $u \in L^p(0,T;X)$ and

$$\sup_{h\in(0,T)} \int_0^{T-h} \frac{\|u(t+h)-u(t)\|_X^p}{h^{\kappa p}} < \infty \text{ or } \sup_{t,\tau\in(0,T);t\neq\tau} \frac{\|u(t)-u(\tau)\|_X}{|t-\tau|^{\kappa}} < \infty \text{ if } p=\infty.$$

Note that for $\kappa = 1$ and $p > 1$ it is equivalent to the space $W^{1,p}(0,T;X)$.

then the solution is unique in sense of trajectories. In addition, if initial data belongs to $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ then we have uniqueness of weak solution on the whole time interval provided that $r \in [11/5, 5/2)$, which is the main novelty presented in [**B4**].

Moreover, we do not only have improved integrability, but we can also get the following regularity result.

Theorem 2.3. *Let the maximal monotone r-graph $\mathcal{A}$ satisfy* (2.52)*, $r \geq 11/5$ and $\mathbf{b} \in N^{1/r',r'}(0,T;W^{-1,r'}_{\boldsymbol{n},\mathrm{div}})$ and $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$. Assume that $t_0 \in (0,T)$ be arbitrary. Then*

$$\boldsymbol{v} \in N^{\frac{1}{2},\infty}(t_0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap N^{\frac{1}{r},r}(t_0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}) \cap N^{\frac{1}{2},2}(t_0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}). \quad (2.61)$$

The same conclusion holds for $t_0 = 0$ provided that $\boldsymbol{v}_0 \in W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ and

$$\sup_{h\in(0,T)} \frac{1}{h}\int_0^h \|\mathbf{b}\|^{r'}_{W^{-1,r'}_{\boldsymbol{n},\mathrm{div}}} < \infty.$$

It is important to notice here that we do not require (2.53) and we also do not require an additional spatial regularity of the right hand side $\mathbf{b}$. In addition, the above result holds in the whole time interval $(0,T)$ provided that data are sufficiently regular. At this point, we do not claim that the assumption $\boldsymbol{v}_0 \in W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ is optimal/sharp, but is really "close to be optimal". Indeed, if $r \geq 12/5$, one can test by time derivative to obtain (2.57) provided that $\boldsymbol{v}_0 \in W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ (otherwise the estimate is not true!). On the other hand, in Theorem 2.3 we obtained that $\boldsymbol{v} \in N^{1/r,r}(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})$ which is of course not embedded into $L^\infty(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})$ but is embedded into any $L^q(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})$ with $q < \infty$. Therefore, we see that the assumption $\boldsymbol{v}_0 \in W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ is almost dictated by the structure of the problem if we want to get the result of Theorem 2.3.

The result in (2.61) is also optimal from the point of view of obtained regularity. Indeed, we cannot get higher fractional differentiability of the solution than those stated in (2.61). If we want to obtain that $\boldsymbol{v} \in N^{\kappa,\infty}_{loc}(0,T;L^2_{\boldsymbol{n},\mathrm{div}})$ then we have to assume $\boldsymbol{v} \in N^{\kappa,2}_{loc}(0,T;L^2_{\boldsymbol{n},\mathrm{div}})$, which corresponds to the theory for standard parabolic equation. The second information follows automatically from interpolation theory in case that $\boldsymbol{v}$ is a weak solution and $\kappa \leq 1/2$. However, if we want to have a higher estimates, i.e., $\kappa > 1/2$, we also need to improve the information about the fractional regularity of $\partial_t \boldsymbol{v}$, which can be read just from the equation and for which we need (2.53) and therefore, the implicit models are excluded. It is not just because of used methods, which can be well documented by standard estimate $\partial_t \boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^d)$, which can be obtained provided that $\operatorname{div} \mathbf{S}(0) \in L^2(\Omega)^d$ but this requires not only that $\boldsymbol{v}_0 \in W^{2,2}(\Omega)^d$ but also that $\mathbf{S}$ is a Lipschitz function of $\nabla \boldsymbol{v}$. Nevertheless, for explicit models, we can go beyond (2.61) and obtain the following result from [**B2**].

Theorem 2.4. *Let the maximal monotone r-graph $\mathcal{A}$ satisfy* (2.52)–(2.53), *$r \geq 11/5$ and $\kappa \in (0,1]$ be arbitrary. Assume that $\mathbf{b} \in N^{\kappa,2}_{loc}(0,T;L^2(\Omega)^d)$. Then any local-in-time weak solution $\boldsymbol{v}$ to* (1.1) *satisfies*

$$\boldsymbol{v} \in N^{\kappa,\infty}_{loc}(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap N^{\kappa,2}_{loc}(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}) \cap N^{\frac{2\kappa}{r},r}_{loc}(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}). \tag{2.62}$$

This theorem thus gives an optimal time regularity result in terms of Nikolskii spaces, i.e., compared with (2.62) it holds for $\kappa > 1/2$. However, this generality requires smoothness of $\mathbf{S}$ (assumed in (2.53)) and is only local in time. We believe that it could be extended also to the initial time $t_0 = 0$ but it would require more smoothness of initial condition and also probably more regularity on $\mathbf{S}$ than just (2.53). It is also worth mentioning that Theorem 2.4 is formulated in [**B2**] only for $r > 11/5$. However, if we adopt the method from [**B4**], it can be proven also for $r = 11/5$.

Furthermore, the above result can be used to obtain also the spatial regularity of solution. Indeed, using (2.62) with $\kappa = 1$, we get that $\partial_t \boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^d)$. Therefore, we can move $\partial_t \boldsymbol{v}$ in (1.1) to the right hand side and consider for each $t \in (0,T)$ the problem (1.1) as elliptic (or more precisely generalized steady Stokes problem). For this steady problem, we can then use the result in [72] to obtain the existence of a strong solution on arbitrary interval (t_0,T) with $t_0 > 0$ for any $r \geq 11/5$. This is indeed the expected result and “corrects" and “fills" the gap in the results from [61, 2], where the authors were able to obtain the existence of strong solution only for $r \geq r_0$ with some $r_0 > 11/5$.

2.3.6 New methods in [B4, B2]

There are essentially three new observations in [**B4**, **B2**], which lead to the desired result. The starting point was the result in [**B2**] (Theorem 2.4). It is based on observation that we can test the equation for time differences by time differences of solution. This is always possible if $\boldsymbol{v}$ is a proper test function, i.e., for $r \geq 11/5$. After such testing, the only term, which does not have a proper sign is the convective term. This term can be however iteratively handled by the following iteration scheme (valid for sufficiently regular $\mathbf{b}$ and $\mathbf{S}$) stated in [**B2**]:

Lemma 2.2. *Let $r > 11/5$. Then there exists $\delta > 0$ such that if $\boldsymbol{v} \in N^{\sigma,2}_{loc}(0,T; W^{1,2}_{\boldsymbol{n},\mathrm{div}})$ then $\boldsymbol{v} \in N^{\min\{1,\sigma+\delta\},2}_{loc}(0,T; W^{1,2}_{\boldsymbol{n},\mathrm{div}})$.*

This lemma then can be applied to obtain (2.62). Unfortunately, Lemma 2.2 cannot be used on the whole interval $(0,T)$ and in addition cannot be used for $r = 11/5$. These two drawbacks were removed in [**B4**]. The first improvement was that Lemma 2.2 holds true globally in $(0,T)$ and for constitutive laws satisfying only (2.52) (so the setting of maximal monotone graphs is allowed) provided that the initial condition belongs to $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$, $\mathbf{b}$ is essentially bounded at initial time zero and that $\sigma + \delta \leq 1/2$. The border $1/2$ comes from the origin of the problem and to cross it one would need more regularity about data and about the constitutive law for $\mathbf{S}$, namely at least (2.53). The second improvement is that we were able to cover also the critical case $r = 11/5$. In fact, in order to start the iteration in Lemma 2.2 also for $r = 11/5$ one needs a "better information" than only $\boldsymbol{v} \in L^r(0,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}})$. Here, the analysis however depends on the precise meaning of "better". In case one tries to improve spatial (with respect to x) information, then one immediately needs to face the troubles with the pressure and localization and it seems to be very difficult to obtain improvement up to the boundary $\partial\Omega$. On the other hand, in [**B4**], we used only improvement with respect to time, i.e., we showed that $\boldsymbol{v} \in L^{r+\varepsilon}(0,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}})$, which was enough to start the iteration in Lemma 2.2. This improvement was done by the generalized parabolic version of the reverse Hölder inequality applied to the quantity $\|\boldsymbol{v}(t)\|_{1,r}$, which do not require any localisation with respect to the spatial variable and is therefore proper for quite general boundary conditions.

3 Heat conducting fluids

In this part, we consider the problem (1.1) but completed with the equation for the internal energy e, which is described by (1.14) and $(1.16)_1$, i.e., we have in addition the equation

$$\partial_t e + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}\left(\kappa(e, |\nabla e|)\nabla e\right) = \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}). \tag{3.1}$$

However, to simplify the presentation, we consider the function κ (the heat conductivity) just being the fnction of e, i.e., $\kappa(e, |\nabla e|) := \tilde{\kappa}(e)$. The implicit law (1.12) is then changed to

$$\mathbf{G}(\mathbf{S}, \mathbf{D}, e) = \mathbf{0}, \tag{3.2}$$

which is then identified by the maximal monotone e-parameterized maximal monotone graph $\mathcal{A}(e)$. A generic example, which is a generalization of (2.4) is the following

$$2\nu(|\mathbf{D}|, e)\left(\tau_*(e) + (|\mathbf{S}| - \tau_*(e))_+\right) \mathbf{D} = \left(|\mathbf{S}| - \tau_*(e)\right)_+ \mathbf{S}, \tag{3.3}$$

where τ_* is a continuous nonnegative bounded. Typical example we have in mind is the function ν of the form

$$\nu(\mathbf{D}, e) = \nu_1(e)((1 + |\mathbf{D}|^2)^q + \nu_2(e))^{\frac{r-2}{2q}}. \tag{3.4}$$

Here, ν has the $(r-2)$-growth and ν_i are continuous positive bounded functions and $q > 0$. Note that if $\tau_* \equiv 0$ then (3.3) is the explicit model, where $\mathbf{S}$ is a continuous function of $\mathbf{D}$ and e.

3.1 Summary of available results

Concerning the analysis of the problem (1.1) and (3.1) we need to face now an additional problem, which is the term $\mathbf{S} \cdot \mathbf{D}$ on the right hand side of (3.1). Recalling the fundamental estimate (1.11), we see that the term $\mathbf{S}\cdot\mathbf{D}$ is a priori only an integrable quantity. Due to the non-reflexivity of the Lebesgue space L^1, it is then very difficult to identify the limit in this term, when one tries to construct the solution. In addition, this term is of the so-called critical growth, which in elliptic and parabolic systems can lead to irregularity of the solution or even to a nonexistence of a weak solution, see e.g. [33, 34, 39] or the survey [68].

Notice that if ν is independent of e then one can first solve (1.1) and having given the velocity field, one can then solve the equation (3.1) with given right hand side which belongs to $L^1(Q)$. Such theory for even nonlinear parabolic or elliptic system is available thanks to [11, 12, 10] or see also the monograph [56], where it is fitted to the context of fluid dynamics. Concerning the regularity of the solution for the problems when ν is independent of e, it is just driven by the regularity of the velocity field given by (1.1), which was discussed in the preceding section.

The situation is however completely changed if ν depends on e. Then one cannot split the problem and must treat both equations simultaneously. Concerning the existence of the weak solution, one can follow the main idea of Ladyzhenskaya [47] and for $r \geq (3d+2)/(d+2)$ (which is the subcritical or critical case) prove the existence of a weak solution. The first result of this type was proven in [26] for Dirichlet boundary conditions but can be easily extended also to the Navier boundary condition. Here again it is quite important that one can test (1.1) by the velocity $\boldsymbol{v}$, which corresponds to $r \geq (3d+2)/(d+2)$.

For the supercritical case, the first existence result is due to [30], where the authors considered the viscosity ν depending only on e, i.e., the Newtonian model (3.4) with $r = 2$ and the spatially periodic problem. The essential observation in [30] is that one needs to replace (3.1) by the balance of global energy $E := |\boldsymbol{v}|^2/2 + e$, which has the form

$$\partial_t E + \operatorname{div}((E+p)\boldsymbol{v}) - \operatorname{div}(\kappa(e)\nabla e + \mathbf{S}\boldsymbol{v}) = \mathbf{b} \cdot \boldsymbol{v}. \tag{3.5}$$

The above identity is formally equivalent to (3.1). Indeed, subtracting the balance of the kinetic energy (1.3) from (3.5) one gets (3.1). However, this is only a **formal** manipulation, which can be justified in case that one can use $\boldsymbol{v}$ as a test function in (1.1), or equivalently saying, if the balance of kinetic energy (1.3) holds true. Furthermore, from the physical point of view, the identity (3.5) is just the first law of thermodynamics, which we assume is true in general. The balance of internal energy (3.1) is then just a consequence of the first law provided that $\boldsymbol{v}$ is sufficiently regular. Moreover, in (3.5) the term $\mathbf{S} \cdot \mathbf{D}$ is missing and we do not need to deal with just L^1 quantities, so even the choice of the global energy as an primitive unknown seems to be more convenient from the mathematical point of view. However, and it is not the case with (3.1), we need to introduce the integrable pressure p in order to give the meaning to all terms appearing in (3.5) and also we have the restriction that that $|\boldsymbol{v}|^3$ is an integrable quantity. The first point somehow excludes the Dirichlet boundary conditions from the analysis since the existence of the pressure as an integrable quantity is in this case unknown for lower value of r's. The second restriction on the integrability of $\boldsymbol{v}$ then gives the bound $r > 3d/(d+2)$.

This observation was used in [17] for proving the existence of a solution for all $r > 3d/(d+2)$ for explicit models (3.3)–(3.4) with $\tau_* \equiv 0$. The extension to implicit models (3.3) with general τ_* and $r = 2$ was done quite recently in [66]. So from the point of view of the existence results, one can think that due to the restriction $r \geq 3d/(d+2)$ the theory is “complete".

However, the quite opposite is true if one looks for the regularity of the solution in case that ν depends on e. Besides the standard problems like in Navier–Stokes equations, we must deal here with the term $\mathbf{S} \cdot \mathbf{D}$ on the right hand side of heat equation, which is the term of the critical growth, which can in general parabolic systems lead to a singular solution even in two dimensional setting, see [33]. In addition, even in the subcritical regimes, i.e., if $r > (3d+2)/(d+2)$, the regularity issues have not been solved yet. Furthermore, even if we neglect the convective term, which is the source of a possible singularity in Navier–Stokes equations, then there were any satisfactory theory missing exactly because of the critical term on the right hand side of the heat equation. The only available results are when the viscosity is “almost" independent of e, or more precisely, if it has very small oscillations over some value. We just recall here two recent results about about the regularity of the solution for such models. In two dimensional setting, it was shown in [24] that for the model (3.3)–(3.4) with $r = 2$, $\tau_* = 0$ and ν_1 given, there exists a smooth solution provided that for all $e_1, e_2 \geq 0$ we have $|\nu_1(e_1) - \nu_1(e_2)| \ll 1$. For the same model, but without the smallness restriction, partial regularity (or regularity for small data) was proven in [25]. Furthermore, the idea from [91] can be adapted to obtain the local regularity results for more general form of viscosities, for example as those in (3.4). However, any global result was missing.

3.2 Presented results [B3, B6]

As mentioned above, the issue in the regularity theory in the heat conducting fluid is not only the presence of the convective term but mainly the coupling with the equation for the internal energy and the presence of the term $\mathbf{S}\cdot\mathbf{D}$ in that equation. Since we do not want to complicate the situation by boundary data, we consider here the spatially periodic problem. We take the model with (3.4) and $\tau_* \equiv 0$ and consider two cases. First, when $r = 2$, i.e., the viscosity is only energy dependent and we have the Newtonian fluid, and the second, where $\nu_1(e) \equiv 1$ but $r \in (1, \infty)$ is general. Just to describe the result in one sentence: In the first case, we obtain the L^2 regularity theory for Stokes–Fourier system, and in the second case, we will get the existence of classical solution for two dimensional case. The results are more precisely described below.

3.2.1 L^2 maximal regularity - [B3]

In [**B3**], we neglected the convective term and considered only Newtonian fluids. To avoid the technical difficulties, we consider only the spatially periodic problem. More precisely, we have the system

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \partial_t \boldsymbol{v} - \operatorname{div} 2\left(\nu_1(e)\mathbf{D}(\boldsymbol{v})\right) &= -\nabla p + \mathbf{b}, \\ \partial_t e - \operatorname{div}(\kappa \nabla e) &= 2\nu_1(e)|\mathbf{D}(\boldsymbol{v})|^2 \end{aligned} \tag{3.6}$$

completed by initial conditions $\boldsymbol{v}_0$ and $e_0 \geq 0$ and assuming that $\boldsymbol{v}$ and e are periodic with respect to $\Omega := (0,1)^d$. Here, $\kappa > 0$ is a constant and ν_1 is a continuous functions fulfilling for all $e \in \mathbb{R}_+$

$$0 < C_1 \leq \nu_1(e) \leq C_2 < \infty.$$

Our goal is to show that if $\mathbf{b} \in L^2(0,T; L^2(\Omega)^d)$ then $\boldsymbol{v} \in L^2(0,T; W^{2,2}(\Omega)^d)$ provided that the initial data are sufficiently regular, or to prove even the higher regularity result. This belongs into a class of maximal regularity results that would for our problem read as

$$\mathbf{b} \in L^p(0,T; L^p(\Omega)^d) \implies \boldsymbol{v} \in L^p(0,T; W^{2,p}(\Omega)^d). \tag{3.7}$$

Note that for the constant viscosity, the relation (3.7) is nothing else than the maximal regularity problem for the Stokes system and this result was proven for $p \in (1,\infty)$ by Solonikov, see e.g. [84]. In addition, the same holds true if the viscosity ν_1 depends on (t,x) and is regular enough, e.g., $\nu \in \mathcal{C}^1(Q)$. However, since we do not a priori know the regularity of e we cannot apply this result. The second possible approach is to mimic the strategy of [24]. In the first step, one can improve the integrability of $\mathbf{D}(\boldsymbol{v})$, which is always possible. So if

$$\sup_{e_1,e_2} |\nu_1(e_1) - \nu_1(e_2)| \le \delta \tag{3.8}$$

then it holds

$$\mathbf{b} \in L^{p(\delta)}(0,T;L^{p(\delta)}(\Omega)^d) \implies \boldsymbol{v} \in L^{p(\delta)}(0,T;W^{1,p(\delta)}(\Omega)^d), \tag{3.9}$$

where $p(\delta) > 2$ depends on δ. If $\delta \to 0_+$ then we can chose p arbitrarily, but for general δ we have that $|p(\delta)-2| \ll 1$. Having such an improvement, we can now use the theory for the heat equation $(3.6)_2$ to get

$$e \in W^{1,\frac{p(\delta)}{2}}(0,T;L^{\frac{p(\delta)}{2}}(\Omega)) \cap L^{\frac{p(\delta)}{2}}(0,T;W^{2,\frac{p(\delta)}{2}}(\Omega)).$$

Hence using the standard parabolic embedding, we see that if $p(\delta) > d+2$ then e is continuous. This then allows one to bootstrap the above mentioned estimates and to finally conclude (3.7) for arbitrary $p \in (1,\infty)$. But such a result requires really small oscillations of ν_1.

Next, we present the result from [**B3**], where we do not require the small oscillation condition but rather some structural assumption. Note that we assume in what follows that all data, i.e., e_0, $\boldsymbol{v}_0$ and $\mathbf{b}$ are Ω-periodic. Furthermore, to fix a constant we always look for $\boldsymbol{v}$ and p that has zero mean value over Ω.

Theorem 3.1. *Let $d \ge 2$ and $e_{\min} > 0$. Assume that $\mathbf{b} \in L^2(0,T;L^2(\Omega)^d)$, $e_0 \ge e_{\min}$, $\sqrt{e_0} \in W^{1,2}(\Omega)$ and $\boldsymbol{v}_0 \in W^{1,2}(\Omega)^d$ such that $\operatorname{div} \boldsymbol{v}_0 = 0$ and $\int_\Omega \boldsymbol{v}_0 = 0$. Let $\nu_1 \in \mathcal{C}^{0,1}(0,\infty)$ fulfill for some $\varepsilon > 0$ and all $s \ge e_{\min}$*

$$\left|\frac{\nu_1'(s)}{\nu_1(s)}\right| \le \frac{-1+\sqrt{15}}{s - e_{\min} + \varepsilon}. \tag{3.10}$$

Then there exists a triple $(\boldsymbol{v}, e, p)$ that solves (3.6) such that

$$\begin{aligned}
\boldsymbol{v} &\in L^2(0,T;W^{2,2}(\Omega)^d) \cap W^{1,2}(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;W^{1,2}(\Omega)^d),\\
\sqrt{e} &\in L^\infty(0,T;W^{1,2}(\Omega)),\\
p &\in L^2(0,T;W^{1,2}(\Omega)).
\end{aligned} \tag{3.11}$$

If in addition $\mathbf{b} \in W^{1,2}(0,T;(W^{1,2}(\Omega)^d)^*)$, $\boldsymbol{v}_0 \in W^{2,2}(\Omega)^d \cap W^{1,4}(\Omega)^d$ *and* $\sqrt{e_0} \in W^{2,2}(\Omega) \cap W^{1,4}(\Omega)$ *then*

$$
\begin{aligned}
\boldsymbol{v} &\in W^{1,\infty}(0,T;L^2(\Omega)^d) \cap W^{1,2}(0,T;W^{1,2}(\Omega)^d),\\
\sqrt{e} &\in W^{1,\infty}(0,T;L^2(\Omega)),\\
e &\in L^{\frac{d+1}{d-2}}(0,T;W^{2,\frac{d+1}{d-2}}(\Omega)) \cap W^{1,\frac{d+1}{d-2}}(0,T;L^{\frac{d+1}{d-2}}(\Omega)).
\end{aligned}
\tag{3.12}
$$

The above result can be in two dimensional setting even extended to the following

Theorem 3.2. *Let* $d = 2$ *and* $k \in \mathbb{N}$. *Assume that* $\nu \in \mathcal{C}^{k-1,1}(0,\infty)$, $\boldsymbol{v}_0 \in W^{k,2}(\Omega)^d$ *and* $e_0 \in W^{k,2}(\Omega)^d$ *satisfy all assumptions of Theorem 3.1 and denote* $(\boldsymbol{v}, e, p)$ *the solution found in Theorem 3.1. Then*

$$
\begin{aligned}
\boldsymbol{v} &\in L^{\infty}(0,T;W^{k,2}(\Omega)^d) \cap L^2(0,T;W^{k+1,2}(\Omega)^d),\\
e &\in L^{\infty}(0,T;W^{k,2}(\Omega)) \cap L^2(0,T;W^{k+1,2}(\Omega)).
\end{aligned}
\tag{3.13}
$$

Moreover, if $(\boldsymbol{v}_1, \boldsymbol{e}_1, p_1)$ *and* $(\boldsymbol{v}_2, e_2, p_2)$ *are two solutions corresponding to the same data and fulfilling* (3.13) *with* $k = 1$ *then* $(\boldsymbol{v}_1, \boldsymbol{e}_1, p_1) = (\boldsymbol{v}_2, e_2, p_2)$.

The strength of above results is that the condition (3.10) is **not small oscillation condition**. In fact the viscosity ν_1 can have very large derivative for values of energy e being not far from $e_{\min}$. The only requirement stated in (3.10) is that it tends sufficiently quickly to some value at infinity. To give a prototypic example, we may consider the Arrhenius law

$$\nu_1(s) := \nu_0 \exp(A(s+K)^{-1})$$

with $A, K > 0$. Then (3.10) is satisfied if

$$e_{\min} > 2A - K.$$

So, it is evident that for example if $2A \le K$ then the assumption (3.10) is satisfied for any initial data. Moreover, we see that the possible oscillations of ν_1 fulfilling the above restriction can be estimated by

$$\nu_1(e_{\min}) - \nu_1(\infty) = \nu_0 \exp(1/2)$$

so we have certainly no smallness assumption.

The starting point for the analysis is Theorem 3.1, while Theorem 3.2 can be understood as the consequence of the first result. Indeed, the most important are the results (3.11)–(3.12), which can be further used to obtain the results of Theorem 3.2 by classical two dimensional methods. However, the classical approach fails when one tries to obtain (3.11). To illustrate it, let us take the scalar product of $\Delta \boldsymbol{v}$ and $(3.6)_2$ and integrate over Ω to get[14]

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 &+ 2\int_\Omega \nu_1(e)|\nabla \mathbf{D}(\boldsymbol{v})|^2\,dx \\ &\le \|\mathbf{b}\|_2\|\Delta \boldsymbol{v}\|_2 + 2\int_\Omega |\nu_1'(e)||\nabla e||\mathbf{D}(\boldsymbol{v})||\nabla \mathbf{D}(\boldsymbol{v})|\,dx,\end{aligned}$$

which directly leads to

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \int_\Omega \nu_1(e)|\nabla \mathbf{D}(\boldsymbol{v})|^2\,dx \le \|\mathbf{b}\|_2^2 + \int_\Omega \frac{|\nu_1'(e)|^2}{\nu_1(e)}|\nabla e|^2|\mathbf{D}(\boldsymbol{v})|^2\,dx. \quad (3.14)$$

It is clear that the last term is now the most problematic one, which cannot be handled from the standard estimates for the heat equation. However, we observed in **[B3]** that the term on the right hand side of (3.6), i.e., the term with the critical growth that is supposed to cause main difficulties, can be surprisingly used for improving the estimates, provided that we choose the **correct** quantity. The correct quantity is already suggested in Theorem and instead of e, we rather deal with $\sqrt{e}$ and consequently with renormalized solutions for the heat equation. This procedure then leads to the statement of Theorem 3.11 provided that the viscosity fulfills (3.10).

3.2.2 Classical solutions for generalized Navier–Stokes–Fourier system - [B6]

Here, we again consider spatially periodic problem and we restrict ourselves only to two dimensional setting, i.e., $\Omega := (0,1)^2$, but consider more general model than in preceding subsection. It means that we include the convective term into our consideration and we have in mind the model (3.4) with $\nu_1(e) \equiv$

[14]We use integration by parts, the periodicity of solution and also the Hölder inequality.

$\nu_0 > 0$. The governing equations then have the form

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0,\\
\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div} 2\left(\nu_0((1+|\mathbf{D}|^2)^q + \nu_2(e))^{\frac{r-2}{2q}}\mathbf{D}(\boldsymbol{v})\right) &= -\nabla p,\\
\partial_t e + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}(\kappa\nabla e) - 2\nu_0((1+|\mathbf{D}|^2)^q + \nu_2(e))^{\frac{r-2}{2q}}|\mathbf{D}(\boldsymbol{v})|^2 &= 0
\end{aligned}
\tag{3.15}
$$

In [**B6**], we consider even for more general relationships between $\mathbf{S}$, $\mathbf{D}$ and e, but to simplify the presentation we restrict ourselves to the prototypic example in (3.15). Also, one could easily add the source term $\mathbf{b}$ to the right hand side of $(3.15)_2$.

The problems in analysis of (3.15) are basically the same as for (3.6). We again have the problem with the critical growth, which even in two dimensional setting may lead to the singularity. Consequently, the existence of a classical solution was not known whenever the viscosity depends on the energy. Nevertheless, and similarly as in the preceding section, we observed that a certain structure of the viscosity leads to the regularity.

Theorem 3.3. *Let $r > 4/3$ and $q \geq r/4$. Assume ν_2 is a smooth function that for some $\varepsilon > 0$ and all $e \geq 0$ satisfies*

$$
|\nu_2'(e)| \leq \frac{C}{(1+e)^{\frac{1}{2}+\varepsilon}}. \tag{3.16}
$$

Then for any Ω-periodic $\boldsymbol{v}_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ and nonnegative $e_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ with $\alpha > 0$, there exists a solution $(\boldsymbol{v}, e, p)$ to (3.15) *fulfilling*

$$
\boldsymbol{v} \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}), \quad \nabla p \in \mathcal{C}^{\alpha,1+\frac{\alpha}{2}}(\overline{Q}), \quad e \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}). \tag{3.17}
$$

Moreover, the solution is unique in the class (3.17).

Here, the function spaces $\mathcal{C}^{\beta,\gamma}(\overline{Q})$ denotes the space where for $\beta,\gamma \geq 0$ fulfilling $\beta = n_1 + \alpha_1$ and $\gamma = n_2 + \alpha_2$ with $n_i \in \mathbb{N}_0$ and $\alpha_i \in [0,1]$

$$
f \in \mathcal{C}^{\beta,\gamma}(\overline{Q}) \quad \Leftrightarrow \quad \nabla^{n_1} f \in \mathcal{C}^{0,\alpha_1}(\overline{Q}) \text{ and } \partial_{t^{n_2}}^{n_2} f \in \mathcal{C}^{0,\alpha_2}(\overline{Q}).
$$

The above result is an consequence of the following theorem from [**B6**].

Theorem 3.4. *Let $\Omega := (0,1)^2$ and $Q := (0,T) \times \Omega$. Assume that $\mathsf{S} : \mathbb{R}_+ \times \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is Lipschitz mapping, which satisfies for some $r \in (4/3, \infty)$ and some positive constants ν_0, ν_1, ν_2 and ε and all $(e, \mathsf{D}, \mathsf{B}) \in \mathbb{R}_+ \times \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$*

$$\begin{aligned} \left|\frac{\partial \mathsf{S}(e,\mathsf{D})}{\partial e}\right| &\le \frac{\nu_2(1+|\mathsf{D}|^2)^{\frac{r-2}{4}}}{(1+e)^{\frac{1}{2}+\varepsilon}}, \\ \nu_0(1+|\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 \le \frac{\partial \mathsf{S}(e,\mathsf{D})}{\partial \mathsf{D}} &\cdot (\mathsf{B}\otimes\mathsf{B}) \le \nu_1(1+|\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2. \end{aligned} \tag{3.18}$$

Then for any Ω-periodic $\boldsymbol{v}_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ and nonnegative $e_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ with $\alpha > 0$, there exists an Ω-periodic solution $(\boldsymbol{v}, e, p)$ to

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, \\ \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div} \mathsf{S}(e, \mathsf{D}(\boldsymbol{v})) &= -\nabla p && \text{in } Q, \\ \partial_t e + \operatorname{div}(e\boldsymbol{v}) - \operatorname{div}(\kappa\nabla e) - \mathsf{S}(e,\mathsf{D}(\boldsymbol{v}))\cdot\mathsf{D}(\boldsymbol{v}) &= 0 && \text{in } Q, \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega, \\ e(0) &= e_0 && \text{in } \Omega \end{aligned}$$

fulfilling

$$\boldsymbol{v} \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}), \quad \nabla p \in \mathcal{C}^{\alpha,1+\frac{\alpha}{2}}(\overline{Q}), \quad e \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}). \tag{3.19}$$

Moreover, the solution is unique in the class (3.19).

Theorem 3.3 is a direct consequence of Theorem 3.4. Indeed, setting

$$\mathsf{S}(e, \mathsf{D}(\boldsymbol{v})) := 2\nu_0((1+|\mathsf{D}|^2)^q + \nu_2(e))^{\frac{r-2}{2q}}\mathsf{D}(\boldsymbol{v})$$

then the assumption (3.16) and the assumption $q \ge r/4$ implies (3.18). All other assumptions of Theorem 3.4 are clearly satisfied.

In the proof of Theorem 3.2, we combined basically two methods. First, we adapted and generalized the two dimensional method invented in [42] to the setting of heat conducting fluids and we include the classical theory for the heat equation. Second, and this is the key step, we provided the uniform estimates for the second velocity gradient. However, due to the presence of e we must again face the difficulties with the critical term on the right hand side of the heat equation. In this case, it is again solved by the structural assumption (3.18), which allows us to get the a priori estimate. Similarly as in the Newtonian case, when one take the scalar product of the momentum equation and $\Delta \boldsymbol{v}$ then after integration over Ω and integration by parts then it follows[15]

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \int_\Omega \frac{\partial \mathbf{S}(e,\mathbf{D}(\boldsymbol{v}))}{\partial \mathbf{D}} \cdot (\partial_k \mathbf{D}(\boldsymbol{v}) \otimes \partial_k \mathbf{D}(\boldsymbol{v}))\, dx \\ = -\int_\Omega \left(\frac{\partial \mathbf{S}(e,\mathbf{D}(\boldsymbol{v}))}{\partial e} \otimes \nabla e \right) \cdot \nabla \mathbf{D}(\boldsymbol{v})\, dx,\end{aligned}$$

which by using (3.18) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \nu_0 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\nabla \mathbf{D}(\boldsymbol{v})|^2\, dx \\ \le \nu_2 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}} |\nabla \mathbf{D}(\boldsymbol{v})| \frac{|\nabla e|}{(1+e)^{\frac{1}{2}+\varepsilon}}\, dx.\end{aligned}$$

Hence, by the Young inequality, we obtain

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + c_1 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\nabla \mathbf{D}(\boldsymbol{v})|^2\, dx \le c_2 \int_\Omega \frac{|\nabla e|^2}{(1+e)^{1+2\varepsilon}}\, dx. \tag{3.20}$$

Next, since from the energy estimate we have that $\int_Q \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}) < \infty$, we can use the theory for the heat equation[16] with L^1 right hand side to obtain

$$\int_Q \frac{|\nabla e|^2}{(e+1)^{1+2\varepsilon}}\, dx \le C(\varepsilon^{-1}).$$

Thus, we see that the right hand side of (3.20)is finite and therefore we can start with the uniform estimate, which allows us to bootstrap and to obtain (3.19).

[15]The convective term vanishes due to the two dimensional setting.
[16]Here the presence of convective term does not change the estimates.

4 Kolmogorov model for turbulent flows [B5]

In this part, we extend the model (1.1) by the description of a turbulent behaviour of the fluid, i.e., we add to (1.1) the system $(1.16)_2$. We do not aim to describe all possible models that are used for description of the turbulent regimes in fluids but we rather focus on the two-equations Kolmogorov model of turbulence invented in [44], see also the English translations [85, Appendix] and [45, paper No. 48]. The model takes the form

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.1}$$

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - 2\nu_0 \operatorname{div}\left(\frac{b}{\omega}\mathbf{D}(\boldsymbol{v})\right) = -\nabla p, \tag{4.2}$$

$$\partial_t \omega + \operatorname{div}(\omega \boldsymbol{v}) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega}\nabla \omega\right) = -\kappa_2 \omega^2, \tag{4.3}$$

$$\partial_t b + \operatorname{div}(b \boldsymbol{v}) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega}\nabla b\right) = -b\omega + \kappa_4 \frac{b}{\omega}|\mathbf{D}(\boldsymbol{v})|^2. \tag{4.4}$$

Similarly as in (1.1), the first two equations of the system (4.1)–(4.4) represent the generalized Navier–Stokes equations for unsteady flows of an incompressible fluid but here the velocity $\boldsymbol{v}$ is just the averaged velocity. The generalization concerns the non-constant effective kinematic viscosity b/ω proposed by Kolmogorov, where b denotes $3/2$ of the turbulent kinetic energy (i.e., the part of the kinetic energy of the fluid that is not described by $|\boldsymbol{v}|^2/2$) and ω the "frequency" of the turbulent pulsations that are of variable length scales ℓ ranging between two positive values (inner and outer scale). The quantity ℓ is linked to ω and b through the relation $\omega = c\sqrt{b}/\ell$, where $c > 0$ is a dimensionless positive constant. The evolution of b and ω is described by the last two equations in which

$$\mu := \frac{b}{\omega}$$

stands for the effective diffusivity of these turbulent characteristics. The parameters $\nu_0, \kappa_1, \ldots, \kappa_4$ appearing in (4.1)–(4.4) are assumed to be positive dimensionless constants; Kolmogorov specified κ_2 to be $7/11$ and considered $\kappa_4 = 2\nu_0$.

To complete the system (4.1)–(4.4) we need to specify the initial and boundary data. Regarding the initial conditions, we assume that

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \qquad b(0) = b_0, \qquad \omega(0) = \omega_0 \qquad \text{in } \Omega. \tag{4.5}$$

Next, we specify the boundary conditions. To simplify the presentation, we introduce a prototypic boundary conditions, which are much less general than those considered in [**B5**]. Nevertheless, they are general enough to generate the turbulent energy. Hence, we assume that (recall here $\Gamma := (0,T) \times \partial\Omega$)

$$b = b_\Gamma \qquad \text{on } \Gamma \tag{4.6}$$

$$\omega = \omega_\Gamma \qquad \text{on } \Gamma \tag{4.7}$$

and the Navier slip boundary condition[17] for $\boldsymbol{v}$, see (1.8).

4.1 Reformulation of the problem & main difficulties

The system (4.1)-(4.4) consists of the generalized Navier–Stokes equations coupled with two scalar evolutionary convection-diffusion equations and the quantity $\mu := b/\omega$ measures the effective kinematic viscosity and the effective diffusivity of turbulence. For the initial frequency ω_0 and boundary data ω_Γ it seems reasonable to assume that they are uniformly positive and bounded from above and then the structure of (4.3) implies that ω remains uniformly positive and bounded. However, b is required to be merely positive initially (which together with the structure of the equation for b implies that b is at least nonnegative in Q). Consequently, μ might degenerate and it is not a priori evident that one can control spatial derivatives of $\boldsymbol{v}$, b and ω. Similarly, as in Section 1, we can formally obtain (compare with (1.4))

$$\|\boldsymbol{v}(t)\|_2^2 + \int_0^t \int_\Omega \frac{4\nu_0 b}{\omega} |\mathbf{D}(\boldsymbol{v})|^2 \, dx \, d\tau + 2\gamma_* \int_0^t \int_{\partial\Omega} |\boldsymbol{v}|^2 \, dS \, d\tau = \|\boldsymbol{v}_0\|_2^2, \tag{4.8}$$

which is deduced from (we again recall (1.3))

$$\partial_t |\boldsymbol{v}|^2 + \operatorname{div}((2p + |\boldsymbol{v}|^2)\boldsymbol{v}) - \operatorname{div}\left(\frac{4\nu_0 b}{\omega}\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\right) + \frac{4\nu_0 b}{\omega}|\mathbf{D}(\boldsymbol{v})|^2 = 0. \tag{4.9}$$

But this estimate does not imply that $\mathbf{D}(\boldsymbol{v})$ belongs to $L^2(Q)$. Similarly, although we sill control

$$\int_0^T \int_\Omega b |\nabla w|^2 \, dx \, dt < \infty \tag{4.10}$$

[17]In fact at this point we make a huge simplification and we refer to [**B5**] for the general setting, based on the results presented in [20, 19].

we do not get that $\nabla\omega$ is integrable again due to the presence of possibly vanishing b. Nevertheless, thanks to (4.8), the last term in (4.4) is integrable and therefore we have certain control on ∇b. On the other hand, this last term in (4.4) is just an L^1 quantity, which is very difficult to handle (it is very similar situation as in heat conducting fluids problem discussed in the previous section). This suggest us to reformulate (4.3) and (4.4) in the following way.

Instead of (4.3) we consider

$$\partial_t\omega + \operatorname{div}(\omega\boldsymbol{v}) - \kappa_1 \operatorname{div}\left(\frac{\nabla(b\omega) - \omega\nabla b}{\omega}\right) = -\kappa_2\omega^2. \tag{4.11}$$

The main advantage in (4.11) is that we do not need to control $\nabla\omega$ but the control of ∇b together with (4.10) gives the equation (4.11) good meaning.

To handle the right hand side of (4.4), we mimic the procedure as for the heat conducting fluid, where we replaced (3.1) by (3.5). Thus, here we set

$$E := \frac{1}{2}|\boldsymbol{v}|^2 + \frac{2\nu_0}{\kappa_4}b, \tag{4.12}$$

multiply (4.4) by ν_0/κ_4, take the scalar product of $\boldsymbol{v}$ and (4.2), and finally sum the resulting identities to arrive at

$$\partial_t E + \operatorname{div}\left(\boldsymbol{v}(E+p)\right) - 2\nu_0 \operatorname{div}\left(\frac{\kappa_3 b}{\kappa_4\omega}\nabla b + \frac{b}{\omega}\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\right) + \frac{2\nu_0}{\kappa_4}b\omega = 0. \tag{4.13}$$

The key advantage of (4.13) is that all difficult terms are in the divergence form.

To conclude, within the context of regular enough solution, the system (4.1)–(4.4) is equivalent to the system consisting of (4.1), (4.2), (4.11) and (4.13). Within the context of weak solutions, the equivalence of (4.4) and (4.13) is true provided that the velocity can be used as a test function in (4.2) or more precisely, if (4.9) holds true. Again similarly as for the heat conducting fluids if one requires that a weak solution to (4.1), (4.2), (4.11) and (4.13) in addition satisfies

$$\partial_t b + \operatorname{div}(b\boldsymbol{v}) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega}\nabla b\right) \geq -b\omega + \kappa_4\frac{b}{\omega}|\mathbf{D}(\boldsymbol{v})|^2, \tag{4.14}$$

in a weak sense, then it is natural to call such a solution **suitable weak solution**. Indeed, subtracting (4.14) from (4.13), one deduces that

$$\partial_t|\boldsymbol{v}|^2 + \operatorname{div}\big((|\boldsymbol{v}|^2+2p)\boldsymbol{v}\big) - \operatorname{div}\left(4\nu_0\frac{b}{\omega}\mathbf{D}(\boldsymbol{v})\boldsymbol{v}\right) + 4\nu_0\frac{b}{\omega}|\mathbf{D}(\boldsymbol{v})|^2 \le 0, \quad (4.15)$$

which is the usual notion of suitable weak solution to the Navier–Stokes system (see [22]).

4.2 Assumptions on the data

In this subsection, we specify our requirements on the data. In particular, our goal is to cover the natural case (with the only assumption of bounded energy) and we also want to include the possibility that the turbulent kinetic energy is not uniformly positive initially.

Thus, we first specify the requirements on the initial data. For the velocity $\boldsymbol{v}$ and the turbulent energy b we assume that

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\operatorname{div}}, \quad (4.16)$$

$$b_0 \in L^1(\Omega),\ b_0 > 0 \text{ a.e. in } \Omega,\ \ln b_0 \in L^1(\Omega). \quad (4.17)$$

Next, for the frequency ω, we assume that there exist $0 < \omega_{\min} \le \omega_{\max} < \infty$ such that

$$\omega_0 \in L^\infty(\Omega) \quad \text{and} \quad \omega_{\min} \le \omega_0 \le \omega_{\max} \qquad \text{a.e. in } \Omega. \quad (4.18)$$

Concerning the boundary conditions for b and ω, we simplify the situation by assuming that ω_Γ and b_Γ can be extended onto the whole of Q (and we denote these extensions again by ω_Γ and b_Γ) such that

$$\omega_\Gamma \in L^{\beta_\Gamma}(0,T;W^{1,\beta_\Gamma}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega)) \text{ for some } \beta_\Gamma > \frac{16}{5}, \quad (4.19)$$

$$b_\Gamma \in L^2(0,T;W^{1,2}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega)). \quad (4.20)$$

In addition, we require that, for $\omega_{\min}$, $\omega_{\max}$ introduced above and for some $0 < b_{\min} \le b_{\max} < \infty$,

$$\omega_{\min} \le \omega_\Gamma \le \omega_{\max} \qquad \text{a.e. in } Q, \quad (4.21)$$

$$b_{\min} \le b_\Gamma \le b_{\max} \qquad \text{a.e. in } Q. \quad (4.22)$$

4.3 Main result

In order to simplify the presentation of the key result and its proof (but not lose any of the generality of the main theorem) we assume in what follows that all material constants $2\nu_0, \kappa_1, \ldots, \kappa_4$ are equal to one. For the same reason we also introduce μ to be defined through

$$\mu := \frac{b}{\omega}, \tag{4.23}$$

and we recall that E, the total kinetic energy, is then defined as

$$E = \frac{|\boldsymbol{v}|^2}{2} + b \tag{4.24}$$

and we also define the proper set of possible b's as

$$\begin{aligned} \mathcal{E} := \{b \in L^1(Q);\ b > 0 \text{ a.e. in } Q, \qquad & b, \ln b \in L^\infty(0,T;L^1(\Omega)), \\ (b - b_\Gamma) \in L^q(0,T;W_0^{1,q}(\Omega)) & \text{ for all } q \in [1,2)\}. \end{aligned} \tag{4.25}$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a $\mathcal{C}^{1,1}$ domain and $T > 0$. Assume that the initial data satisfy (4.16)–(4.18) and the boundary data satisfy (4.19)–(4.22). Then, there exists a quintuple $(\boldsymbol{v}, b, \omega, p, \boldsymbol{s})$ such that*

$$\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}) \cap W^{1,q}(0,T;W^{-1,q}_{\boldsymbol{n}}) \textit{ for all } q \in \left[1, \frac{16}{11}\right), \tag{4.26}$$

$$b \in \mathcal{E}, \qquad (\mathcal{E} \textit{ is defined in } (4.25)) \tag{4.27}$$

$$\partial_t b \in \mathcal{M}(0,T;W_\Gamma^{-1,1}(\Omega)), \tag{4.28}$$

$$p \in L^1(0,T;L^1_0(\Omega)), \tag{4.29}$$

$$E \in W^{1,q}(0,T;W_0^{-1,q}(\Omega)) \textit{ for all } q \in \left[1, \frac{80}{79}\right), \tag{4.30}$$

$$\omega \in L^\infty(0,T;L^\infty(\Omega)), \tag{4.31}$$

$$\partial_t \omega \in L^q(0,T;W_0^{-1,q}(\Omega)) \textit{ for all } q \in \left[1, \frac{16}{11}\right), \tag{4.32}$$

$$b(\omega - \omega_\Gamma) \in L^q(0,T;W_0^{1,q}(\Omega)) \textit{ for all } q \in \left[1, \frac{16}{11}\right), \tag{4.33}$$

$$\omega_{\min} e^{-T\omega_{\max}} \le \omega \le \omega_{\max} \textit{ a.e. in } Q. \tag{4.34}$$

In addition, the pressure p can be decomposed as $p = p_1 + p_2 + p_3$, where

$$p_1 \in L^q(0,T; L^q_0(\Omega)) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{4.35}$$

$$p_2 \in L^{\frac{5}{3}}(0,T; L^{\frac{5}{3}}_0(\Omega)), \tag{4.36}$$

$$p_3 \in L^2(0,T; L^2_0(\Omega) \cap L^\infty_{loc}(\Omega)) \tag{4.37}$$

and the quintuple $(\boldsymbol{v}, b, \omega, p, \boldsymbol{s})$ satisfies the following identities:

$$\begin{aligned} &\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle + \int_\Omega (\mu \mathbf{D}(\boldsymbol{v}) - \boldsymbol{v}\otimes\boldsymbol{v}) \cdot \nabla \boldsymbol{w} \, dx\, dt + \gamma_* \int_\Gamma |\boldsymbol{v}|^2 \, dS\, dt \\ &\qquad = \int_Q p \operatorname{div} \boldsymbol{w} \, dx\, dt \qquad \text{for all } \boldsymbol{w} \in L^\infty(0,T; W^{1,\infty}_{\boldsymbol{n}}), \end{aligned} \tag{4.38}$$

$$\begin{aligned} &\int_0^T \langle \partial_t E, z\rangle \, dt + \int_Q (-\boldsymbol{v}(E+p) + \mu \nabla b + \mu \mathbf{D}(\boldsymbol{v})\boldsymbol{v}) \cdot \nabla z \, dx\, dt \\ &\qquad = -\int_Q b\omega z \, dx\, dt \qquad \text{for all } z \in L^\infty(0,T; W^{1,\infty}_0(\Omega)), \end{aligned} \tag{4.39}$$

$$\begin{aligned} &\int_0^T \langle \partial_t \omega, z\rangle \, dt + \int_Q \left(-\boldsymbol{v}\omega + \frac{\nabla(b\omega)}{\omega} - \nabla b\right) \cdot \nabla z \, dx\, dt \\ &\qquad = -\int_Q \omega^2, z \; dt \qquad \text{for all } z \in L^\infty(0,T; W^{1,\infty}_0(\Omega)). \end{aligned} \tag{4.40}$$

The initial data are attained in the following sense

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\omega(t) - \omega_0\|_2 + \|b(t) - b_0\|_1 = 0. \tag{4.41}$$

Moreover, the following inequality holds:

$$\begin{aligned} &\langle \partial_t b, z\rangle + \int_Q (\mu \nabla b - \boldsymbol{v} b) \cdot \nabla z \, dx\, dt \geq \int_Q (-b\omega + \mu |\mathbf{D}(\boldsymbol{v})|^2) z \, dx\, dt \\ &\qquad \text{for all } z \in \mathcal{C}(0,T; W^{1,\infty}_0(\Omega)) \text{ and } z \geq 0 \text{ a.e. in } Q. \end{aligned} \tag{4.42}$$

Here, the identities (4.38)–(4.40) are weak formulations of (4.2), (4.11) and (4.13). The inequality (4.42) states that we found a suitable weak solution. The attainment of boundary data for b is encoded in the definition of the set $\mathcal{E}$, see (4.25), and the attainment of the trace for ω is understood in the sense of (4.33).

4.4 New methods & tools

We conclude this section by summarizing the key features and main difficulties when analyzing (4.1)–(4.4).

Phenomenological models of turbulence describe flows in terms of averaged quantities (time, spatial or stochastic mean values). It has been conjectured by many scientists, see for example Bardos or Titi [3, 4], that such flows should be regular. Their conjecture is supported by the analysis of a simplified Smagorinsky model of turbulence for which the long-time and large-data well-posedness as well as some higher differentiability of the solution are known, see Ladyzhenskaya [48] or Pares [75], while the full regularity (or more precisely even $C^{1,\alpha}$-regularity) is an interesting open question (even when neglecting the inertia or time-derivative of $\boldsymbol{v}$). While in the Smagorinsky model the relationship between the Cauchy stress and $\mathsf{D}(\boldsymbol{v})$ is nonlinear, in the Kolmogorov model the relation between the Cauchy stress tensor and the velocity gradient is **linear**; the generalized viscosity depends however in a specific manner on two scalar quantities b and ω.

The main aim of [**B5**] was to establish long-time and large-data existence theory for the Kolmogorov two-equation model of turbulence in the spirit of Leray [52], Hopf [41] and Caffarelli, Kohn, Nirenberg [22] (long-time and large-data existence of suitable weak solution). The existence result established here opens the door to the study of regularity properties of such solutions. The scaling of the Navier–Stokes equations plays an important role in the investigation of (partial) regularity associated with the weak solution of the Navier–Stokes equations. Not only does the Kolmogorov system share the same scaling but in fact there is a two-parameter family of scales in which the involved quantities are invariant. More precisely, if $(\boldsymbol{v}, p, \omega, b)$ solves the Kolmogorov system (4.1)-(4.4), then, for any a, b and $\theta > 0$, the quadruple $(\boldsymbol{v}_\theta, p_\theta, \omega_\theta, b_\theta)$, defined through

$$\boldsymbol{v}_\theta(t,x) := \theta^{a-b}\boldsymbol{v}(\theta^a t, \theta^b x), \qquad p_\theta(t,x) := \theta^{2(a-b)}p(\theta^a t, \theta^b x),$$
$$\omega_\theta(t,x) := \theta^{a}\omega(\theta^a t, \theta^b x), \qquad b_\theta(t,x) := \theta^{2(a-b)}b(\theta^a t, \theta^b x),$$

solves the Kolmogorov system as well. Moreover, since the above introduced scaling makes the Kolmogorov model essentially different from the Navier–Stokes system, as the scaling parameters a and b can be chosen independently, one can speculate that the local–in–time or small-data regularity result for the Kolmogorov model (in the spirit of results in [52, 32, 22, 51]) could directly imply even the full regularity for any suitable weak solution. Such qualitative analysis forms a new challenging open problem.

In order to establish the long-time and large-data existence of a suitable weak solution to the initial and boundary-value problem associated with the Kolmogorov PDE system (4.1)-(4.4) we have to overcome several difficulties. First, the effective diffusivity of turbulence and the effective kinematic viscosity b/ω could degenerate, which does not allow one to guarantee the integrability of $\nabla\omega$. Using the relation $b\nabla\omega = \nabla(b\omega) - \omega\nabla b$ and the fact that the quantities on the right-hand side are integrable, we found a reformulation of the equation for ω where we could take the limit. Although we do not control the spatial gradient of ω, we achieved the compactness of ω via a generalization of the Div-Curl lemma (see [71, 87, 88], [31]). Second, the L^2-integrability of $\nabla \boldsymbol{v}$ has been observed by choosing the proper test function in the equation for b. Third, the presence of an L^1-nonlinearity in (4.4) is overcome by replacing it by the equation for $b + |\boldsymbol{v}|^2/2$, which however requires that the pressure is integrable, which is very similar idea to heat conducting fluids.

Up to our knowledge, there is no comparable global in time and large data existence result. There is only an alternative study by Mielke and Naumann, announced in [67] but their approach is different in several aspects. They consider merely the spatially periodic problem, and instead of (4.4) they only proved the inequality (4.14). They also have a stronger assumption on b_0. In our approach, we investigate flows in bounded domains with the turbulence generated on the boundary. The equivalent formulation of the equation for b proposed here does not require one to introduce a measure into our setting, but requires the integrability of the pressure. We show that an integrable pressure exists. Referring also to [23, Section 4], we are not aware of any other result concerning long-time and large-data (or well-posedness) existence of (weak) solutions for a two-equation model of turbulence.

The very similar situation was in case of the one equation model of turbulence. If we assume a priori that quantity $\ell := \sqrt{b}/\omega$ is given then it follows that the equation for ω is redundant and the Kolmogorov system (4.1)–(4.4) reduces to ($k := 3b/2$)

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\left(\nu(k)\mathbf{D}(\boldsymbol{v})\right) &= -\nabla p, \\ \partial_t k + \operatorname{div}(k\boldsymbol{v}) - \operatorname{div}\left(\mu(k)\nabla k\right) &= -\varepsilon(k) + \nu(k)|\mathbf{D}(\boldsymbol{v})|^2, \end{aligned} \tag{4.43}$$

where

$$\nu(k) \sim \sqrt{k}, \quad \mu(k) \sim \sqrt{k} \text{ and } \varepsilon(k) \sim \sqrt{k}k.$$

This is the model obtained by Prandtl [76]. A general drawback of a one-equation model of turbulence, such as that proposed by Prandtl, is that the length scale of the turbulence has to be known a priori.

The global in time and the large data existence theory for this model was developed in [16] and it fact it was a basis to solve the problem stated in the thesis. For the mathematical theory in the spirit of Theorem 4.1, for further features related to this system and references regarding the analysis, numerical computations and some applications (further details and more references can be found in a more recent book [23]). From the point of view of mathematical analysis of initial and/or boundary-value problems relevant to the Navier–Stokes system with the viscosity depending on other scalar quantity/quantities, we recall several works on analysis of problems related to or motivated by (4.43) that were established prior to [23], see [53, 54, 55, 7, 8, 6, 38].

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